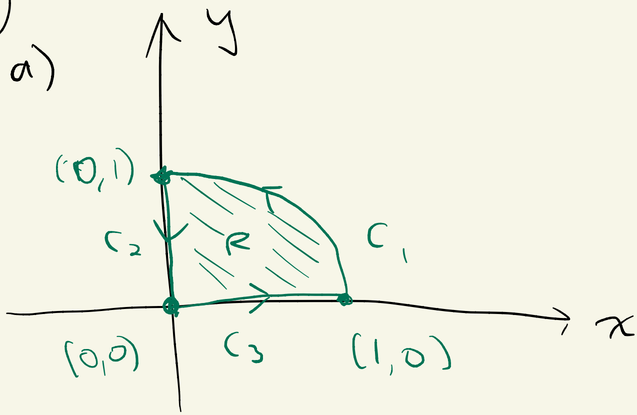
1

a)



b) Use the curl/component test:

$$\text{curl}\,\vec{F} = \hat{k}\left(\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial x}(x)\right)$$

$$= \hat{k}(1-1)$$

$$= \vec{0}$$

$\boxed{\vec{F} \text{ is conservative}}$

$$\text{curl}\,\vec{G} = \hat{k}\left(\frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial x}(x)\right)$$

$$= -2\hat{k}$$

$$\neq \vec{0}$$

$\boxed{\vec{G} \text{ is not conservative}}$

1c) i)

$$\oint_C (\vec{F}+\vec{G})\cdot d\vec{r} = \cancel{\oint_C \vec{F}\cdot d\vec{r}} + \oint_C \vec{G}\cdot d\vec{r}$$

0 since $\vec{F}$ conservative

$$= \oint_C \vec{G}\cdot d\vec{r}$$

$$= \int_{C_1} \vec{G}\cdot d\vec{r} + \int_{C_2} \vec{G}\cdot d\vec{r} + \int_{C_3} \vec{G}\cdot d\vec{r}$$

- $$\int_{C_1} \vec{G}\cdot d\vec{r} = \int_{t=0}^{t=\frac{\pi}{2}} (-\sin t\,\hat{i} + \cos t\,\hat{j}) \cdot \underbrace{(-\sin t\,\hat{i} + \cos t\,\hat{j})}_{=\vec{r}\,'(t)}\,dt$$

parametrize by $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$
$0 \le t \le 2\pi$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t + \cos^2 t \; dt$$

$$= \frac{\pi}{2}$$

1c) i), cont.

Parametrize

$x = 0$
$dx = 0$
$y = 1-t$
$dy = -dt$
$0 \le t \le 1$

- $$\int_{C_2} \vec{G} \cdot d\vec{r} = \int_{C_2} M\,dx + N\,dy$$
$$= \int_0^1 M(0) + N(-dt)$$
$$= \int_0^1 x(-dt)$$

$x = 0$

$$= \int_0^1 -(0)\,dt$$
$$= 0$$

$x = t$
$dx = dt$
$y = 0$
$dy = 0$
$0 \le t \le 1$

- $$\int_{C_3} \vec{G} \cdot d\vec{r} = \int_{C_3} M\,dx + N\,dy$$
$$= \int_0^1 -0 \cdot dt + x \cdot 0$$
$$= 0$$

$$\Rightarrow \boxed{\oint_C \vec{G} \cdot d\vec{r} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}}$$

1 c) ii)

$\vec{F}$ is conservative

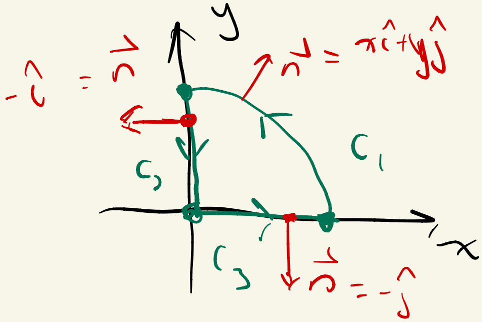
$$\oint_C (\vec{F} - \vec{G}) \cdot d\vec{r} = \cancel{\oint_C \vec{F} \cdot d\vec{r}} - \oint_C \vec{G} \cdot d\vec{r}$$

$$= -\oint_C \vec{G} \cdot d\vec{r}$$

From part i)

$$= -\frac{\pi}{2}$$

1 d) i)



$$\oint_C (\vec{F} + \vec{G}) \cdot \vec{n} \, ds$$

$$= \iint_R \nabla \cdot (\vec{F} + \vec{G}) \, dA$$

divergence Theorem

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} x = 0$

$\nabla \cdot \vec{G} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y} x = 0$

$$= \iint_R \nabla \cdot \vec{F} + \nabla \cdot \vec{G} \, dA$$

$$= \iint_R 0 + 0 \, dA$$

$$\boxed{= 0}$$

ii)

divergence theorem

$$\oint_C (\vec{F} - \pi \vec{G}) \cdot \vec{n} \, ds = \iint_R \nabla \cdot (\vec{F} - \pi \vec{G}) \, dA$$

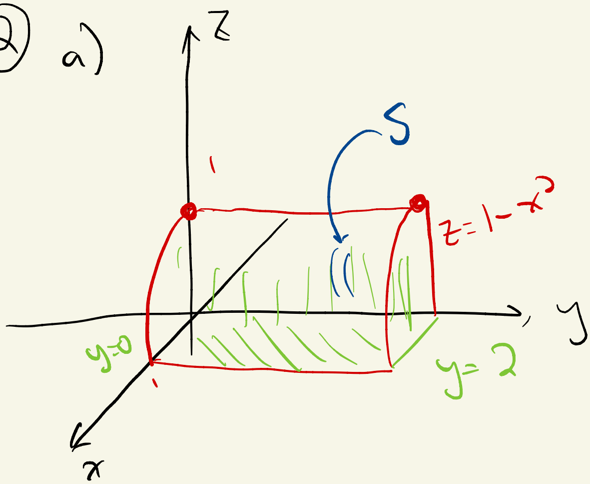
linearity

$$= \iint_R \nabla \cdot \vec{F} - \pi \nabla \cdot \vec{G} \, dA$$

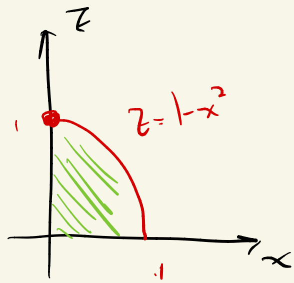
$$= \iint_R 0 - 0 \, dA$$

$$\boxed{= 0}$$

(2) a)



side view:



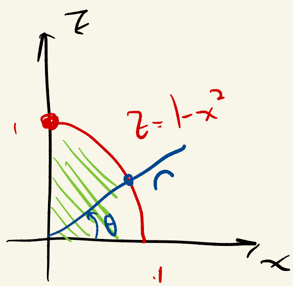
b) i) $\int_{x=0}^{x=1}\int_{y=0}^{y=2}\int_{z=0}^{z=1-x^2} f(x,y,z)\,dz\,dy\,dx$

$$\boxed{= \int_0^1\int_0^2\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx}$$

ii) $x = r\cos\theta$

$y = y$

$z = r\sin\theta$



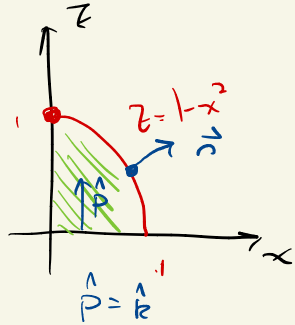
$\int_{y=0}^{y=2}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} f(r,\theta,y)\,r\,dr\,d\theta\,dy$

$$\boxed{= \int_0^2\int_0^{\frac{\pi}{2}}\int_0^1 f(r,\theta,y)\,r\,dr\,d\theta\,dy}$$

c) Let's use implicit method to compute this.

Let $F(x,y,z) = x^2 + z$

$\Rightarrow$ S is the level surface $F(x,y,z) = 1$



$$\vec{n} = \pm \frac{\nabla F}{|\nabla F|}$$

$$= \pm \frac{2x\hat{i} + \hat{k}}{\sqrt{4x^2+1}}$$

to point outward, need $+$!

$$\boxed{= \frac{2x\hat{i} + \hat{k}}{\sqrt{4x^2+1}}}$$

$$d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \hat{p}|}\, dA$$

choose $\hat{p} = \hat{k}$

$$= \frac{\sqrt{4x^2+1}}{|(2x\hat{i}+\hat{k})\cdot\hat{k}|}\, dA$$

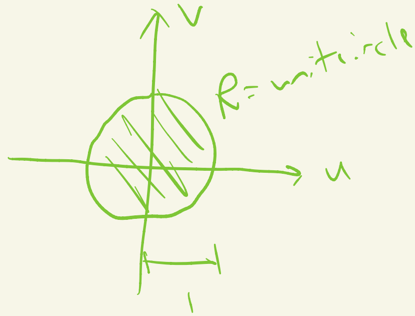
$$\boxed{= \sqrt{4x^2+1}\; dA}$$

(3) a) Let

$x = u$

$y = v$

$z = -\pi x - e y$
$\quad = -\pi u - e v$



$$\Rightarrow \boxed{\begin{array}{c} \vec{r}(u,v) = u\hat{i} + v\hat{j} + (-\pi u - e v)\hat{k} \\ R = \text{unit disc in } (u,v) \end{array}}$$

b) The unit normal to a plane can be found by using

$$\pi x + e y + z = 0$$

$$\Rightarrow \quad \vec{n} = \pm \frac{\langle \pi, e, 1\rangle}{|\langle \pi, e, 1\rangle|} = \pm \frac{\langle \pi, e, 1\rangle}{\sqrt{\pi^2 + e^2 + 1}}$$

want it pointing down

$\Rightarrow$ want $\vec{n} \cdot \hat{k} < 0$

since $\vec{n} \cdot \hat{k} = \pm \frac{1}{\sqrt{\pi^2 + e^2 + 1}}$, choose

$$\boxed{\vec{n} = -\frac{\langle \pi, e, 1\rangle}{\sqrt{\pi^2 + e^2 + 1}}}$$

3c) Let's use explicit form. $f(x,y) = z = -\pi x - e y$

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$= \sqrt{\pi^2 + e^2 + 1}\, dx\, dy$$

$$\Rightarrow \quad \text{Area} = \iint_{\substack{\text{unit}\\ \text{disc}}} d\sigma$$

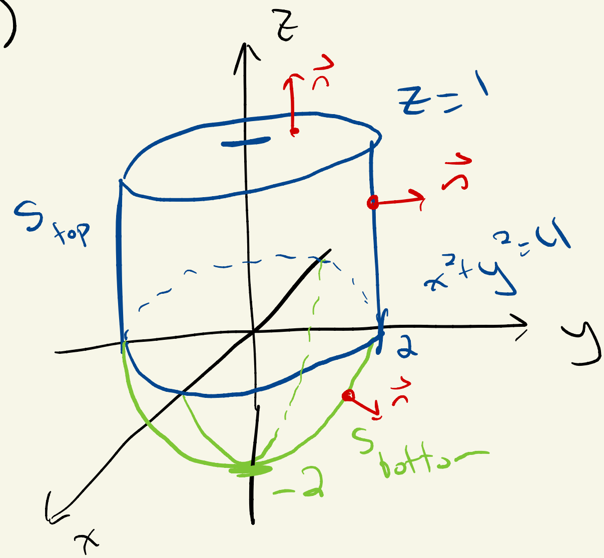
$$= \iint_{\substack{\text{unit}\\ \text{disc}}} \sqrt{\pi^2 + e^2 + 1}\, dx\, dy$$

$$= \sqrt{\pi^2 + e^2 + 1} \iint_{\substack{\text{unit}\\ \text{disc}}} dx\, dy$$

$$= \sqrt{\pi^2 + e^2 + 1}\ \text{Area(unit disc)}$$

$$\boxed{= \sqrt{\pi^2 + e^2 + 1}\ (\pi)}$$

4 a)



b) Let's use the divergence theorem.

$$\oiint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV$$

divergence thm

$$= \iiint_D -1-1-1\,dV$$

$$= -3\iiint_D dV$$

$$= -3\left(\text{Vol}(\text{cylinder}) + \text{Vol}(\text{half sphere})\right)$$

$$= -3\left(\pi(2)^2(1) + \frac{4}{3}\pi(2)^2\right)$$

$$= -12\pi - 16\pi$$

$$\boxed{= -28\pi}$$

4c) i) Note that

$$\iint_{S_{top}} \vec{G}\cdot\vec{n}\,d\sigma = \iint_{S_{top}} (\nabla\times\vec{H})\cdot\vec{n}\,d\sigma$$

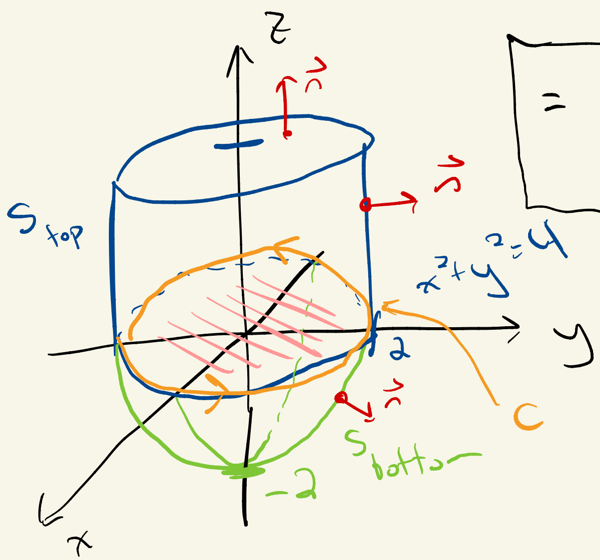
Stokes' thm

$$= \oint_C \vec{H}\cdot d\vec{r}$$

Stokes' again

$$= \iint (\nabla\times\vec{H})\cdot\vec{n}\,d\sigma$$

$$\boxed{= \int_0^{2\pi}\int_0^2 \vec{G}\cdot\hat{k}\; r\,dr\,d\theta}$$



4cii) $\vec{G} = \nabla \times \vec{H}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & e^z \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(1+1)$$

$$= 2\hat{k}$$

$$\iint_{S_{top}} \vec{G} \cdot \vec{n}\, dr = \int_0^{2\pi} \int_0^2 \vec{G} \cdot \hat{k}\; r\,dr\,d\theta$$

i)

$$= \int_0^{2\pi} \int_0^2 2\, r\,dr\,d\theta$$

$$= 2 \int_0^{2\pi} \int_0^2 r\,dr\,d\theta$$

$= 2$ Area(radius 2 disk)

$$= 2 \cdot 2^2 \pi$$

$$\boxed{= 8\pi}$$

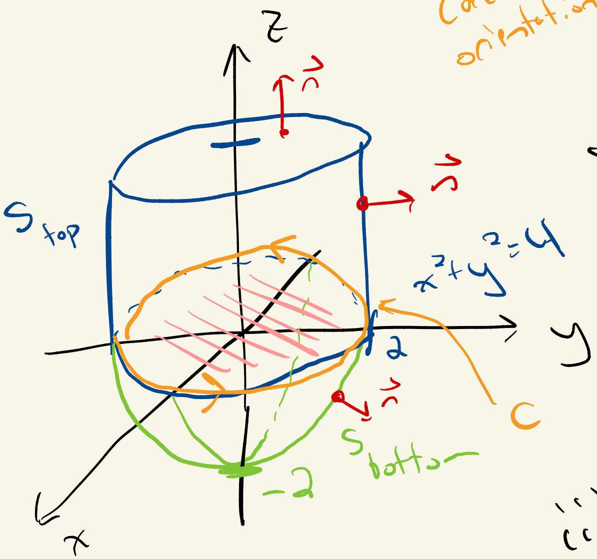
4c iii) $\iint_{S_{bottom}} \vec{G}\cdot\vec{n}\,d\sigma = \iint_{S_{bottom}} (\nabla\times\vec{H})\cdot\vec{n}\,d\sigma$

stokes → $= \oint_{-C} \vec{H}\cdot d\vec{r}$

Careful with orientation!

$= -\oint_{C} \vec{H}\cdot d\vec{r}$

stokes → $= -\iint_{S_{top}} (\nabla\times\vec{H})\cdot d\vec{r}$



$= -\iint_{S_{top}} \vec{G}\cdot\vec{n}$

ii) → $\boxed{= -8\pi}$

4d) "curl of gradient is 0"

$$\oint\!\!\!\!\iint_S \left[\underbrace{\nabla\times\left(\nabla\left(e^{\arctan(xy)}\right)\right)}_{=0}\right]\cdot\vec{n}\,d\sigma$$

$= 0$

5) Let $u = x - 2$

$v = 2y$

$\Rightarrow x = u + 2$

$y = \frac{1}{2} v$

$$\Rightarrow \det(J) = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right)$$

$$= \frac{1}{2}$$

Bounds:

$-\infty < x < \infty$; $-\infty < y < \infty$

$-\infty < x-2 < \infty$; $-\infty < 2y < \infty$

So $-\infty < u < \infty$ and $-\infty < v < \infty$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-((x-2)^2+(2y)^2)}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\left(\frac{1}{2}\right)du\,dv$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\,du\,dv$$

$$\boxed{= \frac{\pi}{2}}$$

6)

a) False.

$$\int_a^b \underbrace{\vec{F}\,|\vec{r}\,'(t)|}_{\text{vector}\cdot\text{scalar} = \text{vector}}\,dt$$

Only integrals of scalar functions give scalars!

b) True. These are integrals that compute arc length of C, which are independent of parameterization.

c) True

By the fundamental theorem of line integrals,

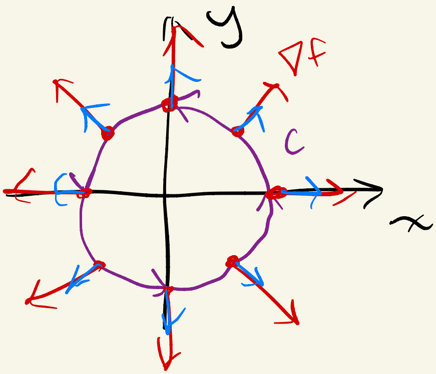
$$\oint_C \nabla f \cdot d\vec{r} = 0 \quad \text{for any loop } C$$

6 d) False. Here's a counterexample.

Suppose $f(x,y) = x^2+y^2$ and suppose C is defined by $1 = x^2+y^2$ (the unit circle).

Then

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$



$$\vec{n} = x\hat{i} + y\hat{j}$$

$$\Rightarrow \oint_C \nabla f \cdot \vec{n}\, ds$$

$$= \oint_C 2(x^2+y^2)\, ds$$

$x^2+y^2=1$

$$= 2\oint_C ds$$

$$= 2\,\text{length}(C)$$

$$> 0$$