Final, Part 1

MAT 21D, Ragone

2021 Summer II

1. Each part of this exam is **40 minutes**. At the end of the 40 minutes, you will have 10 minutes to scan & upload your exam.
2. This exam is open note, open homework, and open book. You may **not** use any external electronic or online resource, nor may you work with others.
3. The final is worth **135 points** total, **68 points** for part 1 and **67 points** for part 2. Where a question has multiple parts, the point breakdown has been listed.
4. You must show **all** your work. Answers without work will receive no credit. Be specific when citing theorems.
5. You may either use physical pen & paper, or an electronic note-taking app. Answer each question on a new page. Please label continuing work by the number and "cont." (e.g., if problem 3 had 3 pages, you could write "3", "3, cont.", "3, cont. 2")
6. Clearly indicate which page goes with which problem. Upload all your pages to GradeScope, and correctly match each question with the pages.
7. Stay logged in to the Zoom room until you have uploaded your exam, in case you need technical assistance. **You do not need to keep your camera on.** If you are disconnected from the Zoom room for any reason, do not worry; message me through Canvas, or send me an e-mail, and continue to work on your midterm until I respond, or until time is up.

This exam is a chance to show off what you have learned so far. If you know how to do part of a question, write down what you know. Good luck!!

Problem 1 (32 points)

Let $\vec{F} = y\hat{\imath} + x\hat{\jmath}$ and $\vec{G} = -y\hat{\imath} + x\hat{\jmath}$ be vector fields on $\mathbb{R}^2$. Let R be the quarter of the unit circle in the first quadrant. Let's call the boundary $\partial R = C$, where $C = C_1 + C_2 + C_3$ is the oriented curve given by traveling counterclockwise along the curve C_1 on the unit circle from $(1,0)$ to $(0,1)$, and then down the straight line C_2 connecting $(0,1)$ to $(0,0)$, and then along the straight line C_3 connecting $(0,0)$ to $(1,0)$.

(a) Sketch the oriented curve C and region R. (**6 points**)

(b) Is $\vec{F}$ conservative? What about $\vec{G}$? (**6 points**)

(c) Compute the line integrals (*Hint: think about b) before computing!*) (**10 points**)

- (i) $\oint_C (\vec{F} + \vec{G}) \cdot d\vec{r}$
- (ii) $\oint_C (\vec{F} - \vec{G}) \cdot d\vec{r}$

(d) Use Green's theorem to compute the following flux integrals (*Hint: recall that the dot product is linear, so* $\nabla \cdot \left(\vec{F} + a\vec{G}\right) = \nabla \cdot \vec{F} + a\nabla \cdot \vec{G}$ *for all real numbers* $a \in \mathbb{R}$.) (**10 points**)

- (i) $\oint_C (\vec{F} + \vec{G}) \cdot \vec{n}\, ds$
- (ii) $\oint_C (\vec{F} - \pi\vec{G}) \cdot \vec{n}\, ds$

Problem 2 (25 points)

Consider the region D where $z \geq 0$ bounded by the parabola $z = 1 - x^2$, the plane $x = 0$, and the planes $y = 0$ and $y = 2$.

(a) Sketch this region. You may add side views to supplement your 3D pictures if it clarifies the image. (**6 points**)

(b) Let $f(x, y, z)$ be a scalar function on $\mathbb{R}^3$. Write a triple integral that computes the volume integral $\iiint_D f(x, y, z)\, dV$ in... (**10 points**)

  (i) ...rectangular coordinates.

  (ii) ...cylindrical coordinates (*hint: you may want to choose the "side cylinder" coordinate system which is polar in xz and has height $y = y$*).

(c) Compute the outward unit normal $\vec{n}$ and the surface area differential $d\sigma$ for section S of the parabola $z = 1 - x^2$ above the xy plane belonging to the boundary ∂D. (**9 points**)

Problem 3 (11 points)

(a) Find a parameterization $\vec{r}(u, v)$ for S, the slice of plane given by the equation $\pi x + ey + z = 0$ lying above the unit disc $x^2 + y^2 \leq 1$. (**4 points**)

(b) Suppose we chose the orientation for this plane such that the unit normal $\vec{n}$ faces in the $-\hat{k}$ direction. Compute $\vec{n}$. (**3 points**)

(c) Write down and compute an integral for the surface area of S. (**4 points**)