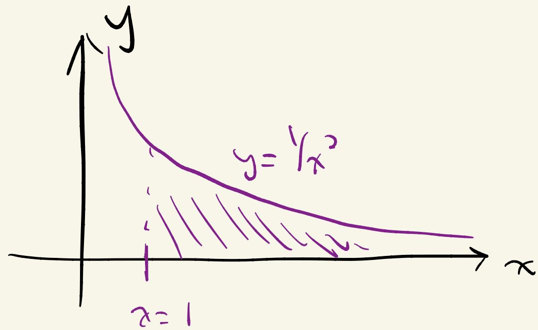
1

i) $\displaystyle\int_1^\infty \int_0^{1/x^2} dy\, dx$



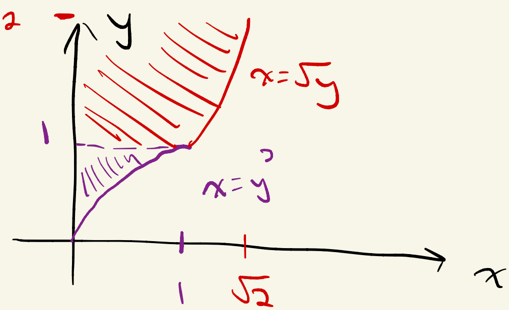
$0 \le y \le 1/x^2$

$1 \le x \le \infty$

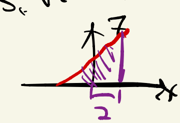
ii) $\displaystyle\int_0^1 \int_0^{y^2} dx\, dy + \int_1^2 \int_0^{\sqrt{y}} dx\, dy$

$0 \le x \le y^2$ $\qquad$ $0 \le x \le \sqrt{y}$

$0 \le y \le 1$ $\qquad$ $1 \le y \le 2$



side view:



iii) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^{r\cos\theta+2} r\, dz\, dr\, d\theta$

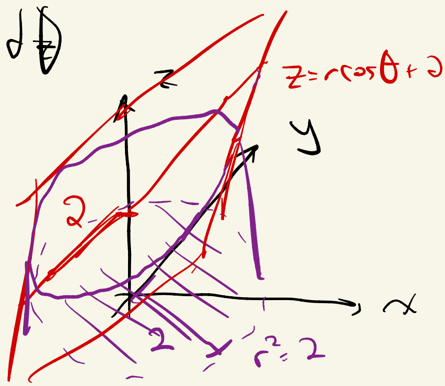
circle radius 2

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$0 \le z \le r\cos\theta + 2$

$= x+2$



(2) $z \geq \sqrt{x^2+y^2}$ $\quad 0 \leq z \leq 1$

with density $\sigma(x,y,z) = x^2+y^2$

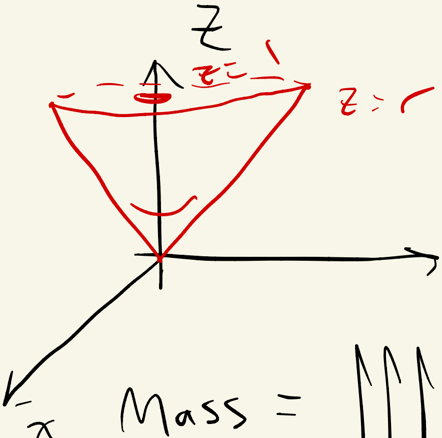
compute center of mass.

Soln:

Saying $z \geq r$ defines cone, $0 \leq z \leq 1$

mass above $(x\,y)$. below $z=1$ we should use cylindrical:

(expect: $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z}$ is between 0 and 1)



$$\text{Mass} = \iiint_R \sigma(x,y,z)\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^z (r^2)\, r\, dr\, dz\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \frac{1}{4} z^4\, dz\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{20}\, d\theta = \boxed{\frac{\pi}{10}}$$

(2), cont.

Both the cone region and the density function $\sigma(x,y,z)$ are symmetric in x on $[-1,1]$ and in y on $[-1,1]$, so

$$\bar{x} = 0$$

$$\bar{y} = 0$$

which leaves only

$$\bar{z} = \frac{1}{M}\iiint_R z\,\sigma(x,y,z)\,dV$$

$$= \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^z z\cdot r^2\cdot r\,dr\,dz\,d\theta$$

$$= \frac{1}{M}\int_0^{2\pi}\int_0^1 z\left[\frac{1}{4}z^4\right]dz\,d\theta$$

$$= \frac{1}{M}\int_0^{2\pi}\frac{1}{4}\cdot\left[\frac{1}{6}z^6\right]_0^1 d\theta$$

$$= \frac{1}{24M}\int_0^{2\pi} 1\,d\theta$$

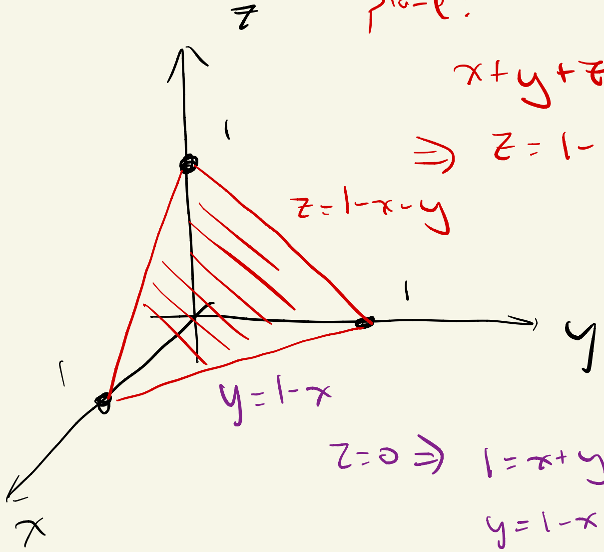
$$= \frac{\pi}{12M} = \frac{\pi}{12}\cdot\frac{10}{\pi} = \boxed{\frac{5}{6}}$$

3

plane:

$x+y+z=1$

$\Rightarrow z=1-x-y$



$z=0 \Rightarrow 1=x+y$

$y=1-x$

3 cont

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} dz\,dy\,dx = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} 1-x-y \; dy\,dx$$

$$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 (1-x)^2 - \frac{1}{2}(1-x)^2 \quad dx$$

$$= \left[-\frac{1}{3}(1-x)^3 + \frac{1}{6}(1-x)^2\right]_{x=0}^{x=1}$$

$$= -\left(-\frac{1}{3} + \frac{1}{6}\right)$$

$$= -\left(-\frac{2}{6} + \frac{1}{6}\right)$$

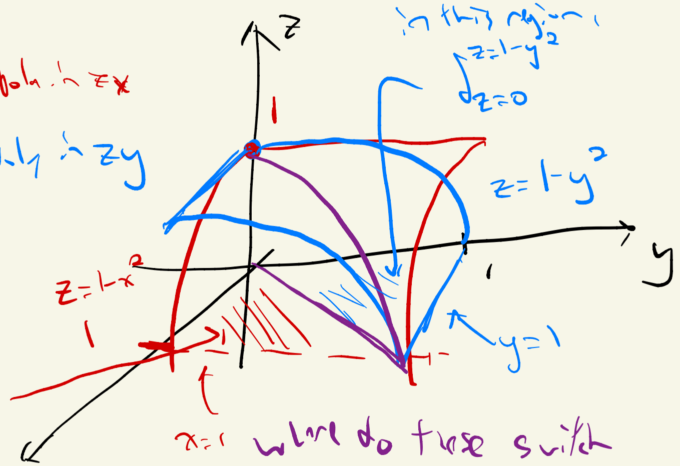
$$\boxed{= \frac{1}{6}}$$

(4)

a) $z = 1 - x^2$ • parabola in zx

$z = 1 - y^2$ • parabola in zy

$x, y, z \geq 0$



where do these switch who's on top?

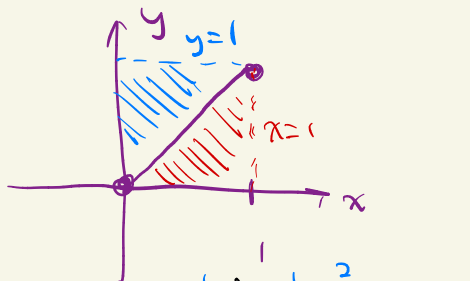
→ • when $z = 1 - y^2$ and $z = 1 - x^2$ intersect. want shadow in xy, so plug in z:

$$\Rightarrow 1 - y^2 = 1 - x^2$$

$$\Rightarrow y^2 = x^2$$

$x \geq 0$ and $y \geq 0$

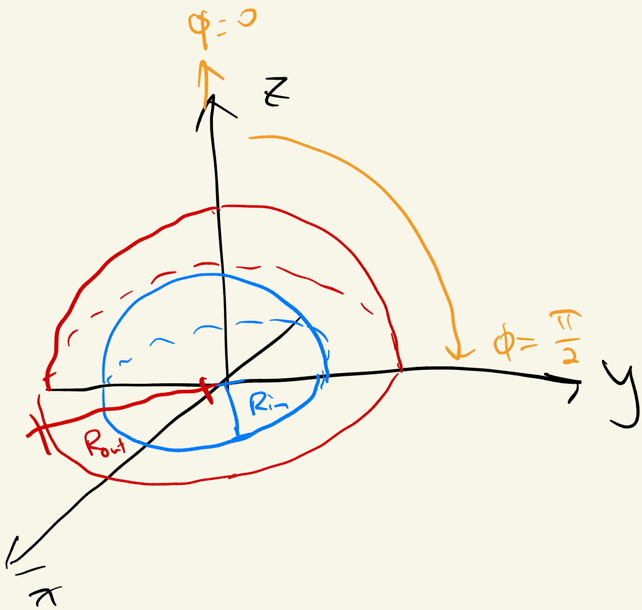
$$\Rightarrow x = y$$



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \int_{z=0}^{z=1-x^2} dz\,dy\,dx + \int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=0}^{z=1-y^2} dz\,dy\,dx$$

$$\boxed{= \int_0^1 \int_0^x \int_0^{1-x^2} dz\,dy\,dx + \int_0^1 \int_x^1 \int_0^{1-y^2} dz\,dy\,dx}$$

4 b)



spherical!

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\rho=R_{in}}^{\rho=R_{out}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\boxed{= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{R_{in}}^{R_{out}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

5 a) We already know the equations of this transformation:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$(x,y) = T(r,\theta)$$

Now, the Jacobian:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

So

$$\det(J) = \cos\theta \cdot r\cos\theta - \sin\theta \cdot (-r\sin\theta)$$

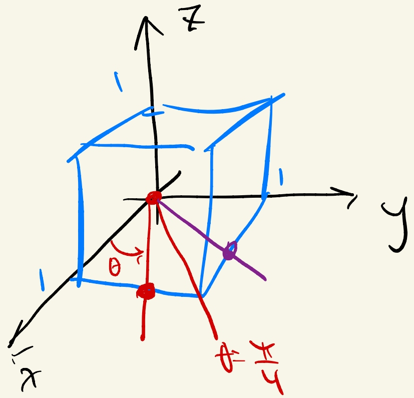
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$\Rightarrow \quad dx\,dy = \det(J)\,dr\,d\theta$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

5b)



• On $0 \le \theta \le \frac{\pi}{4}$, upper bound for r: $x = 1 \Rightarrow r\cos\theta = 1$

$\Rightarrow r = \frac{1}{\cos\theta}$

• On $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, upper bound for r: $y = 1 \Rightarrow r\sin\theta = 1$

$\Rightarrow r = \frac{1}{\sin\theta}$

• z stays the same.

$$\Rightarrow \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=0}^{r=1/\cos\theta} r\,dr\,d\theta\,dz + \int_{z=0}^{z=1} \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1/\sin\theta} r\,dr\,d\theta\,dz$$

$$\boxed{= \int_0^1 \int_0^{\pi/4} \int_0^{1/\cos\theta} r\,dr\,d\theta\,dz + \int_0^1 \int_{\pi/4}^{\pi/2} \int_0^{1/\sin\theta} r\,dr\,d\theta\,dz}$$

6a) <u>True</u>. Note that if a number $q \geq 0$, then $q = |q|$. This means we can rewrite our function σ so that

$$\sigma(x,y,z) = |x^2+y^2| + |z^2|$$

since $x^2+y^2 \geq 0$ and $z^2 \geq 0$

$$= x^2+y^2+z^2$$

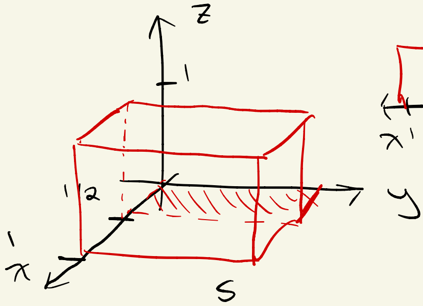
which is a polynomial and thus continuous.

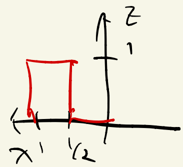
So, Fubini's theorem applies and the written integral is correct.

6b) False. As a counterexample, consider the "step function" on $[0,1]\times[0,1] = S$

$$f(x,y) = \begin{cases} 1 & \text{if } \quad 1/2 \le x \le 1 \\ 0 & \text{if } \quad x < 1/2 \end{cases}$$

which is clearly not continuous.





Then

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 \left[\int_0^{1/2} 0\,dx + \int_{1/2}^1 1\,dx\right) dy$$

$$= \int_0^1 1/2\,dy$$

$$= 1/2$$

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^1 0\,dy\,dx + \int_{1/2}^1 \int_0^1 1\,dy\,dx$$

$$= \int_{1/2}^1 1\,dx$$

$$= 1/2$$

So they are equal!

6 c) False:

This integral computes the volume of a half cylinder, since $\theta = \pi$: