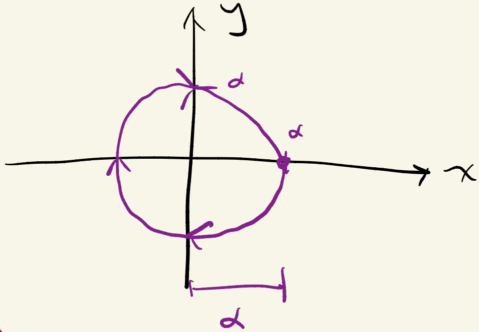
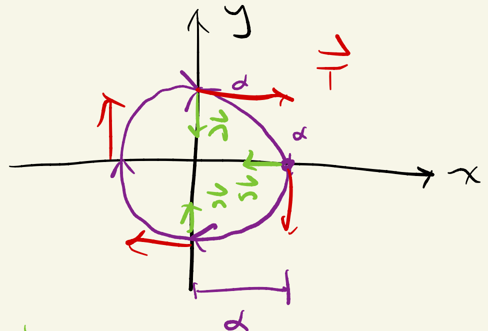
1 a)



b) $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$= \dfrac{-\alpha \sin t\,\hat{i} - \alpha \cos t\,\hat{j}}{\sqrt{\alpha^2 \sin^2 t + \alpha^2 \cos^2 t}}$

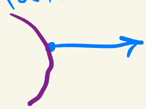


$\alpha > 0 \rightarrow\ = \boxed{\begin{matrix} -\sin t\,\hat{i} - \cos t\,\hat{j} \\ 0 \le t \le 2\pi \end{matrix}}$

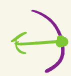
$\vec{n}$ = perpendicular normal to $\vec{T}$

Two choices:

$\vec{n} = \cos t\,\hat{i} - \sin t\,\hat{j}$ ← at $t = 0$, (outward facing)



or $\vec{n} = -\cos t\,\hat{i} + \sin t\,\hat{j}$ ← at $t = 0$ (inward facing)



Here, since CW rotation, $\vec{n}$ should be inward facing

$$\boxed{\begin{matrix} \vec{n} = -\cos t\,\hat{i} + \sin t\,\hat{j} \\ 0 \le t \le 2\pi \end{matrix}}$$

c) Flipping orientation just means swap t with $-t$. Call this new parametrization $\vec{r}_-(t)$.

- $\vec{r}_-(t) = \vec{r}(-t)$

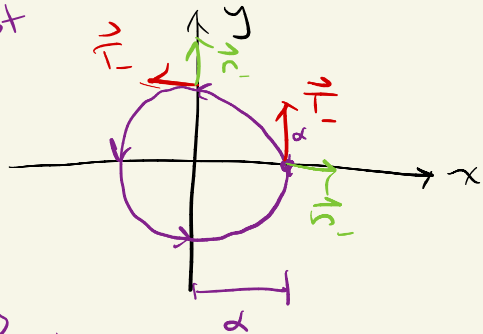
$$= \alpha\cos(-t)\,\hat{i} - \alpha\sin(-t)\,\hat{j}$$

cos is even, sin is odd

$$\boxed{\begin{gathered} = \alpha\cos(t)\,\hat{i} + \alpha\sin(t)\,\hat{j} \\ 0 \le t < 2\pi \end{gathered}}$$

ccw rotation

- Orientation reversing just changes CW rotation to CCW rotation... so we only need to flip signs (see picture): (alternatively recognize this is the usual circle CCW param)



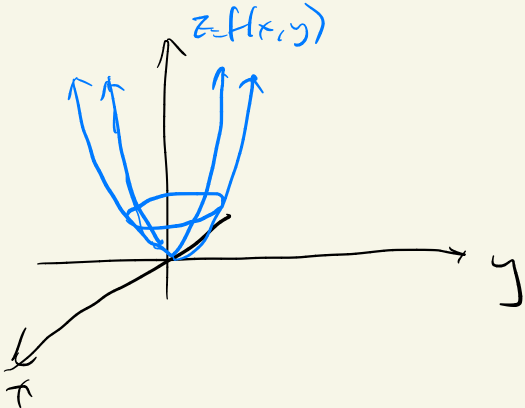
$$\vec{T}_- = \boxed{\begin{gathered} -\sin t\,\hat{i} + \cos t\,\hat{j} \\ 0 \le t < 2\pi \end{gathered}}$$

$$\vec{n}_- = \frac{\vec{r}_-(t)}{|\vec{r}_-(t)|}$$

since it's a circle ccw!

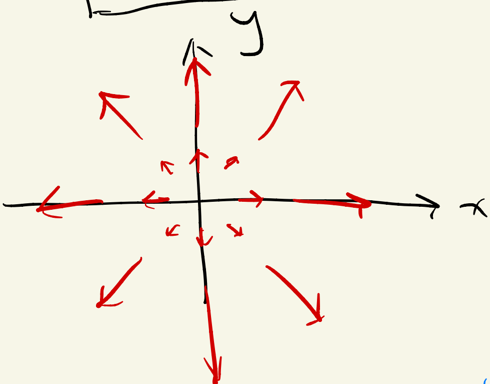
$$\boxed{\begin{gathered} = \cos t\,\hat{i} + \sin t\,\hat{j} \\ 0 \le t < 2\pi \end{gathered}}$$

(2) $f(x,y) = x^2 + y^2$

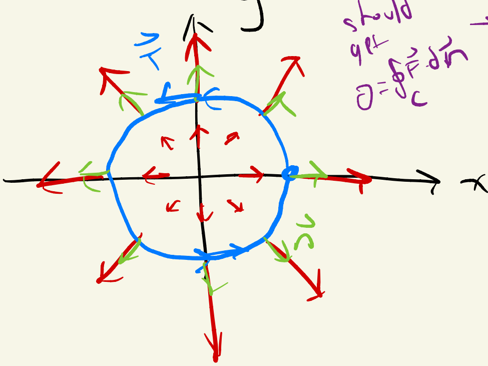


a) $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$\boxed{= 2x\,\hat{i} + 2y\,\hat{j}}$$



b)



should get $0 = \oint_C \vec{F} \cdot d\vec{n}$ →

- Observe:
- $\vec{T}$ is perpendicular to ∇F
- $\vec{n}$ is parallel to ∇F

↑ should get

$$\oint \vec{F} \cdot \vec{n}\, ds = \oint |\vec{F}|\, ds$$

$$= \text{circumference} \cdot |\vec{F}|$$

2 i) By the loop property of gradient vector fields,

$$\oint_C \nabla f \cdot d\vec{r} = 0$$

ii) $\vec{n}$ is outward normal

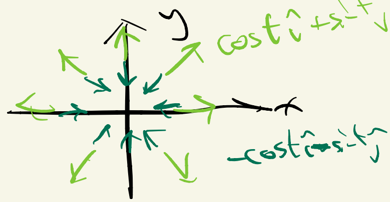
$\Rightarrow$ perpendicular to $\vec{T}$

$$\vec{r}\,'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$$

$\Rightarrow$ if we choose $\vec{n} = \cos t\,\hat{i} + \sin t\,\hat{j}$ ← outward facing

or $-\cos t\,\hat{i} - \sin t\,\hat{j}$ ← inward facing

this will work!



[Alternatively: we showed in class that for a circle,

$\vec{n}$ = unit vector in $\vec{r}$ direction

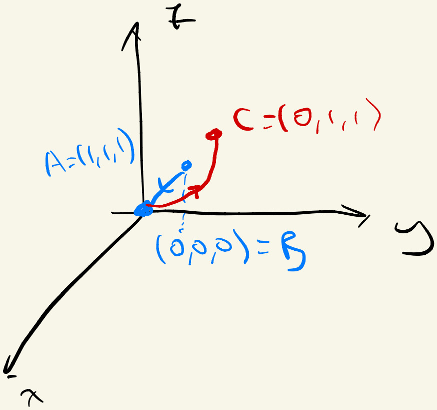
$= \dfrac{\vec{r}}{|\vec{r}|}$]

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} \left(2\cdot 2\cos t\,\hat{i} + 2\cdot 2\sin t\,\hat{j}\right)\cdot\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right) \cdot \left|-2\sin t\,\hat{i} + 2\cos t\,\hat{j}\right| dt$$

$$= \int_0^{2\pi} 4\cdot 2\,dt$$

$$\boxed{= 16\pi}$$

3



a) $\vec{r}_1(t) = \langle 1,1,1\rangle - t\langle 1,1,1\rangle$

$$0 \le t \le 1$$

$$\Rightarrow \int_A^B \vec{F}\cdot d\vec{r}_1 = \int_0^1 \left(-(1-t)\hat{i} + (1-t)\hat{j} + (1-t)\hat{k}\right) \cdot \left(-\hat{i} - \hat{j} - \hat{k}\right) dt$$

$$= \int_0^1 (1-t) - (1-t) - (1-t)\, dt$$

$$= \int_0^1 -(1-t)\, dt$$

$$= -1 + \frac{1}{2}$$

$$\boxed{= -\frac{1}{2}}$$

b) $\vec{r}_2(t) = t\hat{j} + t^3\hat{k}$ (Graph parametrization!)

$0 \leq t \leq 1$

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r}_2 = \int_0^1 \left(-t\hat{i} + 0\hat{j} + t^3\hat{k}\right) \cdot \left(0\hat{i} + \hat{j} + 3t^2\hat{k}\right) dt$$

$$= \int_0^1 3t^5 \, dt$$

$$= 3\left[\frac{t^6}{6}\right]_0^1$$

$$\boxed{= \frac{1}{2}}$$

4

Let's write equations first.

$$\frac{\partial f}{\partial x} \overset{(1)}{=} y\cos(x)$$

$$\frac{\partial f}{\partial y} \overset{(2)}{=} \sin(x)$$

Integrate (1):

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx \overset{(*)}{=} y\sin(x) + g(y)$$

Differentiate:

$$\sin(x) + \frac{\partial g}{\partial y} \overset{(*)}{=} \frac{\partial f}{\partial y} \overset{(2)}{=} \sin(x)$$

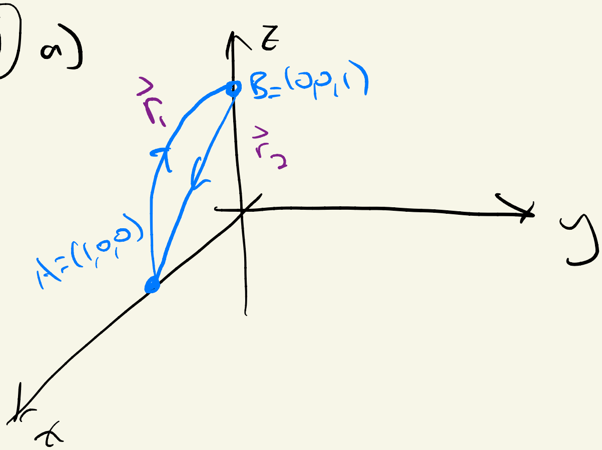
$$\Rightarrow \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow \quad g = c \text{ , constant}$$

$$\Rightarrow \boxed{f(x,y) = y\sin(x) + c}$$

check! $\nabla f = y\cos(x)\,\hat{i} + \sin(x)\,\hat{j}$

5 a)



b) $\vec{r}_1$ is along the circle:

$$\vec{r}_1(t) = \cos t\,\hat{i} + 0\,\hat{j} + \sin t\,\hat{k}$$

$$0 \le t \le \frac{\pi}{2}$$

Quarter circle!

as always, there are lots of other correct answers!

$\vec{r}_2$ is the straight line:

$$\vec{r}_2(t) = \hat{k} + t(\hat{i} - \hat{k})$$

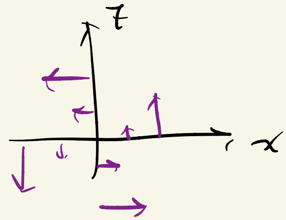
$$= t\,\hat{i} + (1-t)\,\hat{k}$$

$$0 \le t \le 1$$

5ci) $\vec{F} = -z\hat{i} + 0\hat{j} + x\hat{k}$

Note - this vector field is just like the whirlpool in the (x,z) plane.

$\Rightarrow$ Definitely not conservative



$$\oint_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C_1} \vec{F}(\vec{r}_1(t))\cdot \vec{r}_1{}'(t)\,dt}_{(1)} + \underbrace{\int_{C_2} \vec{F}(\vec{r}_2(t))\cdot \vec{r}_2{}'(t)\,dt}_{(2)}$$

$$(1) = \int_0^{\pi/2} \left(-\sin t\,\hat{i} + 0\hat{j} + \cos t\,\hat{k}\right)\cdot\left(-\sin t\,\hat{i} + 0\hat{j} + \cos t\,\hat{k}\right)dt$$

$$= \int_0^{\pi/2} \sin^2 t + \cos^2 t\,dt$$

$$= \frac{\pi}{2}$$

$$(2) = \int_0^1 \left(-(1-t)\hat{i} + 0\hat{j} + t\hat{k}\right)\cdot\left(\hat{i} + 0\hat{j} - \hat{k}\right)dt$$

$$= \int_0^1 -(1-t) - t\,dt$$

$$= \int_0^1 -1 + t - t\,dt$$

$$= -1$$

$$\Rightarrow \boxed{\oint_C \vec{F}\cdot d\vec{r} = \frac{\pi}{2} - 1}$$

5c ii) Recognize that this is a conservative vector field we studied in class:

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \nabla(xyz)$$

$\Rightarrow$ Gradient vector fields have the loop property by fundamental thm of line integrals, so

$$\oint_C \vec{G} \cdot d\vec{r} = 0$$

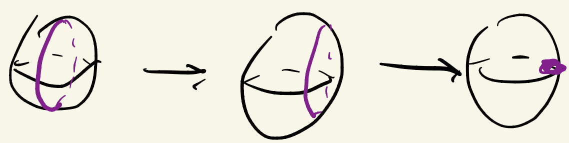
6

a) False.

The problem is the domain $D = \mathbb{R}^2$.
At $(0,0)$, we have a singularity
and $f(x,y)$ is undefined.
(This would be true if we changed D to
e.g. the punctured plane)

b) True.

The component/curl test requires that
$\vec{F}$ has continuous partials on a
simply connected domain. The
sphere S^2 is simply connected (we said this
in class!), since we can contract
any "rubber band" on it to a point:



Since curl $\vec{F} = \vec{0}$, the component test
guarantees $\vec{F}$ is conservative, which
by a theorem in class means $\vec{F}$ has
a potential.

6 c) <u>True</u>.

Use the component/curl test:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & \alpha & \beta \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\beta - \cancelto{0}{\frac{\partial}{\partial z}\alpha}\right)\hat{i}$$

$$- \left(\frac{\partial}{\partial x}\beta - \cancelto{0}{\frac{\partial}{\partial z}M(x)}\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}\alpha - \cancelto{0}{\frac{\partial}{\partial y}M(x)}\right)\hat{j}$$

$$= \vec{0}$$

$\frac{\partial}{\partial z}M = 0 = \frac{\partial}{\partial y}M$

$M = M(x)$

$\Rightarrow$ By the curl test, since $\vec{F}$ has continuous partials, $\vec{F}$ is conservative.