The Curious Symmetry Breaking of $O(n)$ Quantum Spin Chains

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…or, what I did during grad school.

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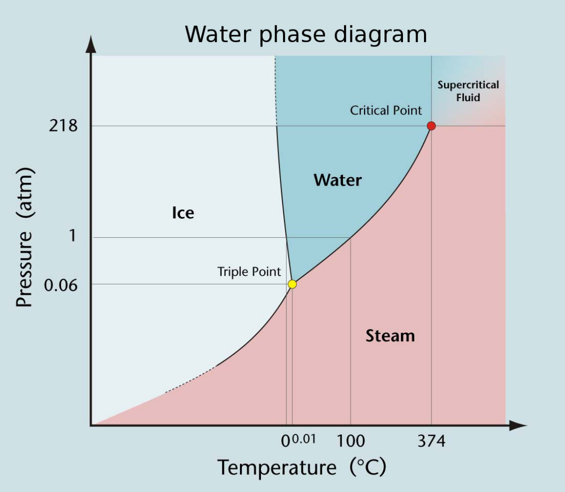
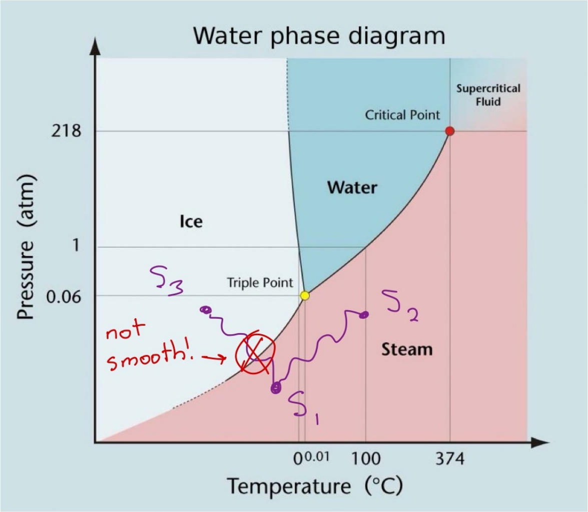
Phases of matter



What is a phase?



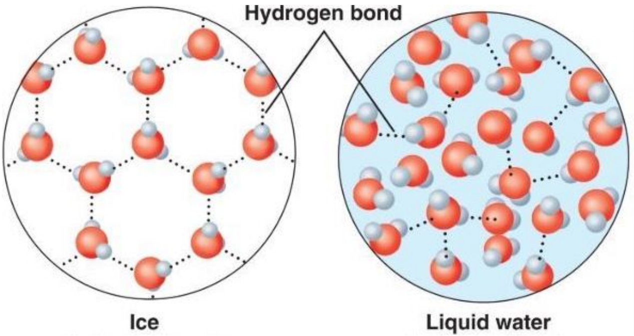
"Two systems are in the same phase if we can find a smooth path connecting them."

Phases and invariants

- How do we distinguish phases?
- One approach: find invariants preserved along paths.
- The "does it have a well-defined volume?" invariant.
    - Water and ice have a well-defined volume.
    - Steam does not.
- This cannot distinguish water from ice.
- By finding more invariants, we can distinguish more phases!

Phases and invariants

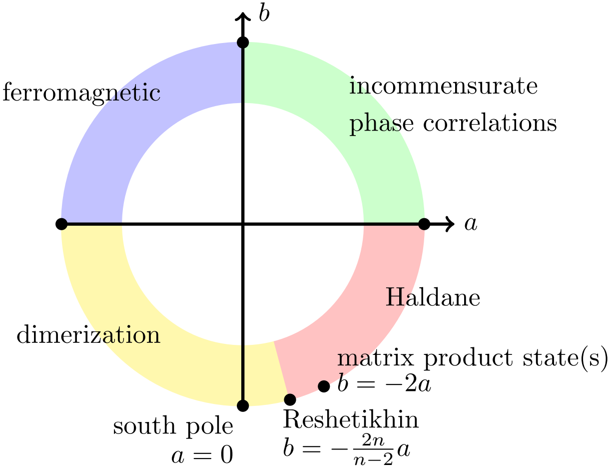
Rich source of invariants: symmetry properties.



Time-averaged density $\rho(x)$, $x \in \mathbb{R}^3$:

- Liquid: $\rho(x) = \rho(x + v)$, $v \in \mathbb{R}^3$. Continuous symmetry.
- Not so for ice! Only has discrete lattice symmetry.

Another phase diagram



Our goal: better understand (and prove the distinctness of) the "Haldane" and "dimerized" phases.

Classical analogy for setup

Suppose we have a particle which moves on the space $[a, b] \subseteq \mathbb{R}$. We want to describe a collection of experiments which measures its position.

- *Observable:* Position function $A(x) = x$.
  - This lives in an algebra,[1] say $\mathcal{A} = C([a, b])$, with some topological structure, say $\|\cdot\|_\infty$.
- *State:* ω is integration against a probability density $\rho(x)dx$, representing the position of a particle taken over many experiments.
- The expected position of the distribution $\rho(x)dx$ is

$$\omega(A) = \int_{[a,b]} x\,\rho(x)dx.$$

Experiment = pairing of observable $A \in \mathcal{A}$ and state $\omega \in \mathcal{A}^*$.

[1] Vector space + multiplication

Finite Quantum spin chains

- Work on finite chains $[a, b] \subseteq \mathbb{Z}$.
- Each site $x \in \mathbb{Z}$ has a Hilbert space $\mathcal{H}_x = \mathbb{C}^n$. Combining sites is tensor product:

$$\mathcal{H}_{[a,b]} = \mathcal{H}_a \otimes \mathcal{H}_{a+1} \otimes \cdots \otimes \mathcal{H}_b.$$

  Bases are easy to find. For example, if $\mathcal{H}_1$ has orthonormal basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ has orthonormal basis

$$\begin{matrix} |00\rangle\,, & |01\rangle\,, \\ |10\rangle\,, & |11\rangle\,, \end{matrix}$$

  where $|ij\rangle = |i\rangle \otimes |j\rangle$.
- So, $\dim(\mathcal{H}_{[1,\ell]}) = n^\ell$.

Finite Quantum spin chains

- *Observables:* $\mathcal{A}_{[a,b]} = \mathcal{B}(\mathcal{H}_{[a,b]})$ with operator norm $\|\cdot\|_\infty$.
  - $\mathcal{A}_{[1,\ell]}$ is just $n^\ell \times n^\ell$ complex matrices $M_{n^\ell}(\mathbb{C})$.
  - Note: this is a noncommutative algebra!
- *States:* positive, normalized linear functionals $\omega : \mathcal{A}_{[a,b]} \to \mathbb{C}$.
  - Since $\dim(\mathcal{H}_{[a,b]})$ finite, Riesz representation theorem applied to the inner product $\langle A, B\rangle = \mathrm{Tr} A^* B$ on $\mathcal{A}_{[a,b]}$ says there is a $\rho \in \mathcal{A}_{[a,b]}$ such that

    $$\omega(A) = \mathrm{Tr} \rho A.$$

    $\rho = \rho^* \geq 0$ is positive $\rho \geq 0$, and ρ is normalized $\mathrm{Tr}\rho = 1$. In this sense ρ is a quantum analogue of a probability density function.
  - *Pure states*: density matrices are rank-1 orthogonal projections $\rho = |\psi\rangle\langle\psi|$ onto the subspace spanned by $|\psi\rangle \in \mathcal{H}_{[a,b]}$.

Quantum spin chain setting

- The "energy" (and thus the dynamics) of the system is dictated by a Hamiltonian $H_{[a,b]} = H^*_{[a,b]} \in \mathcal{A}_{[a,b]}$.
- *Locality*: a particle at site $x \in \mathbb{Z}$ should not interact too strongly with spatially distant $y \in \mathbb{Z}$.
- Nearest neighbor Hamiltonian: $H_\ell \in \mathcal{A}_{[1,\ell]}$

$$H_\ell := \sum_{x=1}^{\ell-1} h_{x,x+1},$$

  where $h_{x,x+1}$ is a copy of $h = h^* \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ acting on sites $x, x+1$.
- We will study ground states. Pure ground states correspond to lowest eigenvalue eigenvectors $H_{[1,\ell]} |\psi\rangle = \lambda |\psi\rangle$.

Example: Just Transverse Fields

- Qubit chain: each site is $\mathcal{H}_x = \mathbb{C}^2$, so Hilbert space is $\mathcal{H}_{[1,\ell]} = \mathbb{C}^{2^\ell}$.
- On-site orthonormal basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$.
- Hamiltonian:

$$H_{[1,\ell]} = \sum_{x=1}^{\ell} Z_x,$$

where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. How to find ground state? Easy here, since $[Z_x, Z_y] = 0$ so each term is simultaneously diagonalizable.

$$H_{[1,\ell]} |11\dots 1\rangle = (Z_1 + Z_2 + \cdots + Z_\ell) |11\dots 1\rangle = -\ell \, |11\dots 1\rangle .$$

Example: The Heisenberg Chain

- Qubit chain: each site is $\mathcal{H}_x = \mathbb{C}^2$, so Hilbert space is $\mathcal{H}_{[1,\ell]} = \mathbb{C}^{2^\ell}$.
- Hamiltonian:

$$H_{[1,\ell]} = \sum_{x=1}^{\ell-1} -\texttt{SWAP}_{x,x+1},$$

where $\texttt{SWAP}_{x,x+1} |u\, v\rangle = |v\, u\rangle$.

- To find ground states, we diagonalize $H_{[1,\ell]}$. Let's start with $\ell = 2$, $\mathcal{H}_{[1,\ell]} = \mathbb{C}^4$,

$$\begin{aligned} \texttt{SWAP} |00\rangle &= |00\rangle \\ \texttt{SWAP} \big(|01\rangle + |10\rangle\big) &= |01\rangle + |10\rangle \\ \texttt{SWAP} \big(|11\rangle\big) &= |11\rangle \\ \texttt{SWAP} \big(|01\rangle - |10\rangle\big) &= -\big(|01\rangle - |10\rangle\big) . \end{aligned}$$

Example: The Heisenberg Chain

- In other words: in the basis $\underbrace{|00\rangle, |10\rangle + |01\rangle, |11\rangle}_{sym}, \underbrace{|01\rangle - |10\rangle}_{antisym}$,

$$\texttt{SWAP} = P_{sym} - P_{antisym} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

  So: $-\texttt{SWAP}$ has pure ground states whose vectors are in the range of P_{sym}.

- Not hard to see that ground state vectors of $H_{[1,\ell]}$ are the symmetric tensors $\mathrm{Sym}^{\ell}(\mathbb{C}^2) \subseteq (\mathbb{C}^2)^{\otimes \ell}$.
- The ground state vector for the interaction $h = \texttt{SWAP}$ is just $|01\rangle - |10\rangle$, which is *entangled*, i.e. cannot be expressed as 1 simple tensor.

Symmetry and Symmetry Breaking

- Symmetry is often given as a unitary representation (cts. group homomorphism $G \to \mathcal{B}(\mathcal{H}_{[1,\ell]})$) of a Lie group.
- Ex: the Heisenberg Chain is invariant under tensor rep of $SU(2)$:

$$[U^{\otimes \ell}, H_{[1,\ell]}] = 0, \qquad \text{for all } U \in SU(2).$$

- Question: do the ground states of H also possess this symmetry?

Symmetry and Symmetry Breaking

Simultaneous Block-Diagonalization Lemma:
Let $U : G \to \mathcal{B}(\mathcal{H}_{[1,\ell]})$ a rep of G and $H = H^*$ such that

$$[U_g, H] = 0 \quad \text{for all } g \in G.$$

Then, for any eigenvector $|\psi\rangle$ with eigenvalue λ, $U_g |\psi\rangle$ is also an eigenvector of H with eigenvalue λ.
Proof:

$$H(U_g |\psi\rangle) = U_g H |\psi\rangle = U_g \lambda |\psi\rangle = \lambda(U_g |\psi\rangle).$$

Consequence: The ground state space is invariant under G.

- If every pure ground state $|\psi\rangle$ is invariant under symmetry, i.e. $U_g |\psi\rangle = |\psi\rangle$ for all $g \in G$, we say the symmetry is *unbroken*.

Symmetry and Symmetry Breaking

- What else might happen?
- Ex: The ground state space $\mathrm{Sym}^2(\mathbb{C}^2)$ of the 2-site Heisenberg chain is invariant under $U \otimes U$, $U \in SU(2)$. But $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2)$ and

$$U \otimes U \left|00\right\rangle = \big(U \left|0\right\rangle\big) \otimes \big(U \left|0\right\rangle\big) = \left|11\right\rangle .$$

  H is invariant under an $SU(2)$ symmetry, but its ground states aren't. The $SU(2)$ symmetry is *broken*.
- Let's introduce the phase diagram.
- Just like the water example, symmetry breaking can be used to distinguish phases.

A phase diagram: $O(n)$-invariant spin chains

- On-site Hilbert space: $\mathcal{H}_x = \mathbb{C}^n$.
- The tensor representation of $O(n)$, the orthogonal group, on $\mathbb{C}^n \otimes \mathbb{C}^n$ is given by $R \mapsto R \otimes R$.
- MAT 261 (Lie Groups) $\implies$ the irrep decomposition[2] is

$$\mathbb{C}^n \otimes \mathbb{C}^n \cong \underbrace{M_2 \oplus \mathbb{C}\,|\xi\rangle}_{sym} \oplus \underbrace{\textstyle\bigwedge^2(\mathbb{C}^n)}_{antisym},$$

  where $|\xi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |ii\rangle$.
- We can then parameterize orthogonally invariant n.n. interactions, i.e. those with

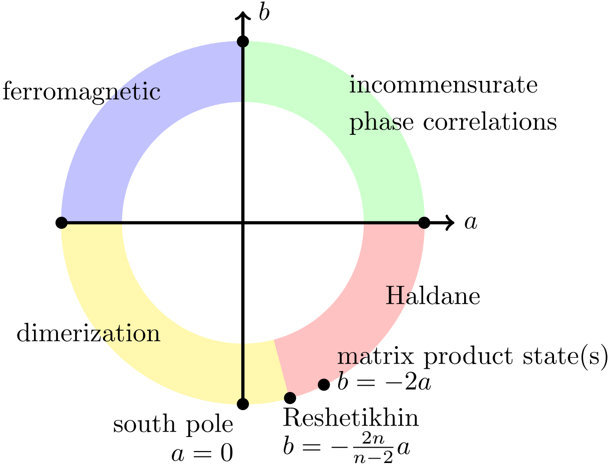
$$(R \otimes R) h_{x,x+1} (R^{-1} \otimes R^{-1}) = h_{x,x+1}.$$

[2] ...except $n = 4$.

A phase diagram: $O(n)$-invariant spin chains

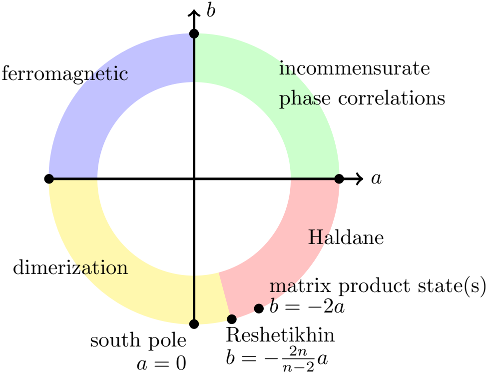
Up to a shift in ground state energy, every such interaction is given by

$$h = a\,\mathtt{SWAP} + b\,|\xi\rangle\langle\xi| + \mathbb{1}, \qquad a, b \in \mathbb{R}.$$



Phases as equivalence classes

"Two model Hamiltonians H_0, H_1 are in the same gapped phase if we can find a smooth path of gapped Hamiltonians connecting them."



How do we distinguish the Haldane phase from the dimerized phase?

Dimerization at the south pole

- Dimerization is a type of symmetry breaking where translation invariance of H is broken by a pair of 2-periodic ground state.
- Physically, dimerization happens when Hamiltonian encourages strong entanglement + "monogamy of entanglement".
- This is what happens at the south pole point: $Q = |\xi\rangle\langle\xi|$, and

$$H_{[1,\ell]} = \sum_{x=1}^{\ell-1} -Q_{x,x+1}.$$

  $-Q_{x,x+1}$ rewards entanglement across sites $x, x+1$. But particles can only entangle strongly with one neighbor.
- Björnberg et.al. 2021 proved there are two 2-periodic ground states $\omega_\pm$ such that:
  - Entanglement structure "alternates"
  - ω_+ and ω_- distinguishable by 2-site observable

The odd n case

- When n is odd, Tu et. al. 2008 showed the matrix product state (MPS) point has only one translation-invariant ground state.
- So: translation symmetry is unbroken in the Haldane phase, but broken in the dimerized phase.

Theorem (Bachmann, Mickalakis, Nachtergaele, Sims 2012)

Two Hamiltonians H_0, H_1 connected by a smooth path of uniformly gapped Hamiltonians must have the same number of pure ground states.

- When n is odd, these phases are distinct.

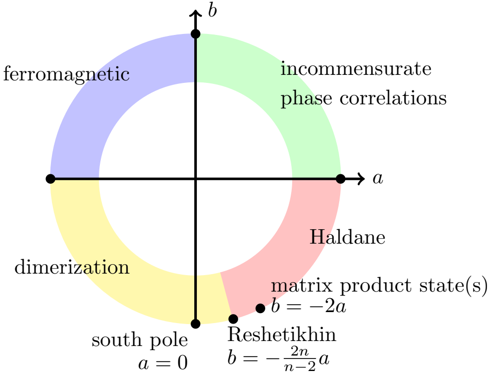
The even n case

- When n is even, the MPS point has a pair of 2-periodic ground states $\omega_{\pm}$. So dimension is not enough.
  - Will later see this is *not* dimerization. Physical mechanism is very different!

- Our lesson from water: studying symmetry properties can help us distinguish these phases. We have a natural $G = O(n)$ symmetry.

G-symmetric phases as equivalence classes

"Two G-symmetric Hamiltonians H_0, H_1 are in the same G-gapped phase[3] if we can find a smooth path of G-symmetric gapped Hamiltonians connecting them."



[3] aka *symmetry-protected topological (SPT) phase*

G-symmetric phases as equivalence classes

- Let G be a compact Lie group (this includes finite groups). Let H_0, H_1 be G-symmetric gapped Hamiltonians. Then let $s = 0, 1$ and define $\mathcal{S}(s) \subseteq \mathcal{A}^*$ to be ground state space of H_s.
- Just like simultaneous block diagonalization: $\mathcal{S}(s)$ is invariant under G.
  - If every $\omega \in \mathcal{S}(s)$ is invariant under G, i.e. $g \cdot \omega = \omega$ for all $g \in G$, then G-symmetry is *unbroken*.
  - If there exists an $\omega \in \mathcal{S}(s)$ where $g \cdot \omega \neq \omega$, then G-symmetry is *broken*.

Theorem (Bachmann, Nachtergaele 2014)

Two G-symmetric Hamiltonians H_0, H_1 connected by a smooth path of uniformly gapped G-symmetric Hamiltonians have that $\mathcal{S}(0) \cong \mathcal{S}(1)$ as G-representations.

$O(n)$-to-$SO(n)$ symmetry breaking

- It was proven in Björnberg 2021 that the south pole ground states $\widetilde{\omega}_+, \widetilde{\omega}_-$ are both invariant under $O(n)$: for all $R \in O(n)$ and $A_1 \otimes \cdots \otimes A_\ell \in \mathcal{A}_{[1,\ell]}$,

$$\widetilde{\omega}_+(RA_1R^{-1} \otimes \cdots \otimes RA_\ell R^{-1}) = \widetilde{\omega}_+(A_1 \otimes \cdots \otimes A_\ell)$$
$$\widetilde{\omega}_-(RA_1R^{-1} \otimes \cdots \otimes RA_\ell R^{-1}) = \widetilde{\omega}_-(A_1 \otimes \cdots \otimes A_\ell).$$

$O(n)$-to-$SO(n)$ symmetry breaking

Theorem (Nachtergaele, R)

The MPS ground states ω_+, ω_- are invariant under $SO(n)$, but swapped under $O(n)$, i.e. for any $R \in O(n)$ with $\det(R) = -1$,

$$\omega_+(RA_1R^{-1} \otimes \cdots \otimes RA_\ell R^{-1}) = \omega_-(A_1 \otimes \cdots \otimes A_\ell).$$

Symmetry is broken! These two points support different irreps of $O(n)$ and so these are different $O(n)$-symmetric phases.

Corollary

The south pole point and the MPS occupy distinct $O(n)$ gapped phases.

The MPS point: it ain't dimerization

- Two key features of dimerization:
  - Entanglement structure "alternates".
  - ω_+ and ω_- are distinguishable by a 2-site observable.
- It was believed that the even n MPS ground states were dimerized. But we found something surprising.

The MPS point: it ain't dimerization

Theorem (Nachtergaele, **R**)

1. *ω_+ and ω_- have identical entanglement structure.*
2. *$\omega_+ \neq \omega_-$, but ω_+ and ω_- are indistinguishable by any observable with support $k < n/2$, i.e. for any $A_1 \otimes \cdots \otimes A_k \in \mathcal{A}_{[1,k]}$,*

$$\omega_+(A_1 \otimes \cdots \otimes A_k) = \omega_-(A_1 \otimes \cdots \otimes A_k).$$

So: if $n \geq 6$, then every 2-local observable has identical expectation:

$$\omega_+(A_1 \otimes A_2) = \omega_-(A_1 \otimes A_2) \qquad \text{for all } A_1, A_2 \in M_n(\mathbb{C}).$$

Let's talk a little representation theory.

Building the MPSs

The interaction $h = \texttt{SWAP} - 2\left|\xi\right\rangle\left\langle\xi\right| + \mathbb{1} \geq 0$ is frustration free, i.e.

$$\ker \sum_{x=1}^{\ell-1} h_{x,x+1} = \bigcap_{x=1}^{\ell-1} \ker h_{x,x+1} \neq \{0\}.$$

Finite chain ground states are given by MPSs.

The rank-n Clifford algebra $\mathcal{C}_n$

The rank-n Clifford algebra is the associative algebra generated by the operators $\gamma_1, \ldots, \gamma_n$ subject to

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}\mathbb{1}, \qquad \gamma_i^* = \gamma_i, \qquad i, j = 1, \ldots, n.$$

- Finite dimensional matrix algebra
- n even: center is just $\mathbb{1}$, so $\mathcal{C}_n = M_{2^{n/2}}(\mathbb{C})$.

Matrix Product States

When n even, $\mathcal{B} = \mathcal{C}_n$. Define MPS by $\psi : \mathcal{B} \to (\mathbb{C}^n)^{\otimes \ell}$

$$\psi(B) = \sum_{i_1,\ldots,i_\ell=1}^{n} \operatorname{Tr}\left(B\gamma_{i_\ell} \ldots \gamma_{i_1}\right) |i_1, \ldots, i_\ell\rangle \,.$$

- Let's get some intuition for $n = 4$.

Let's explicitly compute the MPSs on $\ell = 2$ sites for $n = 4$. Recall that since $h = \texttt{SWAP} - 2\,|\xi\rangle\langle\xi| + \mathbb{1}$, the ground state space is $\mathcal{G}_2$

$$\mathcal{G}_2 = \bigwedge^2 \mathbb{C}^4 \oplus \mathbb{C}\,|\xi\rangle\langle\xi|\,.$$

Clifford algebra basis $\mathcal{B} = \mathcal{C}_4$:

$$\begin{array}{ll} \mathbb{1} & \\ \gamma_i & 1 \le i \le n \\ \gamma_{i_1}\gamma_{i_2} & 1 \le i_1 < i_2 \le n \\ \gamma_{i_1}\gamma_{i_2}\gamma_{i_3} & 1 \le i_1 < i_2 < i_3 \le n \\ \gamma_0 & \end{array}$$

A useful lemma

Lemma

$Tr\gamma_I\gamma_J = 0$ whenever $I \neq J$.

Proof.

(Idea)

$$2\mathrm{Tr}\gamma_1 = \mathrm{Tr}\gamma_1(\gamma_2\gamma_2 + \gamma_2\gamma_2) = \mathrm{Tr}\gamma_2(\gamma_1\gamma_2 + \gamma_2\gamma_1) = 0.$$

□

Calculate some MPSs, with $D = \operatorname{Tr} \mathbb{1}$.

$$\frac{1}{D}\psi(\mathbb{1}) = \frac{1}{D}\sum_{i_1,i_2} \operatorname{Tr}\left(\mathbb{1}\, \gamma_{i_2}\gamma_{i_1}\right)|i_1, i_2\rangle$$

$$=$$

$$\frac{1}{D}\psi(\gamma_1) = \frac{1}{D}\sum_{i_1,i_2} \operatorname{Tr}\left(\gamma_1\, \gamma_{i_2}\gamma_{i_1}\right)|i_1, i_2\rangle$$

$$=$$

$$\frac{1}{D}\psi(\gamma_1\gamma_2) = \frac{1}{D}\sum_{i_1,i_2} \operatorname{Tr}\left(\gamma_1\gamma_2\, \gamma_{i_2}\gamma_{i_1}\right)|i_1, i_2\rangle$$

$$=$$

$$\frac{1}{D}\psi(\gamma_0) = \frac{1}{D}\sum_{i_1,i_2} \operatorname{Tr}\left(\gamma_0\, \gamma_{i_2}\gamma_{i_1}\right)|i_1, i_2\rangle$$

$$=$$

Finite chain ground states

Check: every $\psi(B)$ is a ground state of H.

Finite chains as $SO(n)$ representations

- The Lie algebra $\mathfrak{so}(n)$ embeds into the Clifford algebra in a natural way. Let $L_{i,j} = |i\rangle\langle j| - |j\rangle\langle i|, 1 \leq i < j \leq n$. Then the map $\pi : L_{i,j} \mapsto \frac{1}{2}\gamma_i\gamma_j$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathcal{C}_n$:

$$\frac{1}{4}[\gamma_i\gamma_j, \gamma_r\gamma_s] = \delta_{jr}\gamma_i\gamma_s - \delta_{ir}\gamma_j\gamma_s + \delta_{is}\gamma_j\gamma_r - \delta_{js}\gamma_i\gamma_r. \tag{1}$$

  Since $\mathcal{C}_n \cong M_{2^{n/2}}(\mathbb{C})$, this defines a Lie algebra representation on $\mathbb{C}^{2^{n/2}}$ called the *spin* representation.

- Exponentiates to unitary representation $\Pi : Spin(n) \to \mathcal{U}(\mathbb{C}^{2^{n/2}})$, where we recall $Spin(n)/\{\pm\mathbb{1}\} \cong SO(n)$.
  - If $\Pi(\mathbb{1}) = \Pi(-\mathbb{1})$, this descends to a representation of $SO(n)$.
  - If $\Pi(\mathbb{1}) \neq \Pi(-\mathbb{1})$, only get a *projective* representation of $SO(n)$, i.e.

$$\Pi(w)\Pi(v) = \alpha(w,v)\Pi(wv), \qquad w, v \in SO(n),$$

    where $\alpha : SO(n) \times SO(n) \to U(1)$ is a phase.

Finite chains as $SO(n)$ representations

- Turns out, this is what we need to understand the symmetry of our MPSs:

$$w^{\otimes \ell}\psi(B) = \psi(\Pi(w)B\Pi(w)^{-1}), \qquad B \in \mathcal{B}, w \in SO(n), \tag{2}$$

  where we use that $Spin(n)/\{\pm \mathbb{1}\} \cong SO(n)$.
- This has a nice tensor network representation.
- Thermodynamic limiting states inherit this symmetry! So if we rotate the full chain,

$$\ldots w \otimes w \otimes w \otimes w \otimes \ldots$$

  then ω is invariant under this.
- This reaction to symmetry will shed light on the states and give yet another index.

Finite chains as $SO(n)$ representations

Define $\psi_\ell : \mathcal{B} \to (\mathbb{C}^n)^{\otimes \ell}$, and write $\mathcal{G}_\ell = \{\psi_\ell(B) : B \in \mathcal{B}\}$.

Theorem (Nachtergaele, R)

Let $\ell \geq n$ and let $V = \mathbb{C}^n$. Then the representation $\mathcal{G}_\ell$ of $SO(n)$ decomposes as

$$\mathcal{G}_\ell = \begin{cases} \bigwedge^1 V \oplus \bigwedge^3 V \oplus \cdots \oplus \bigwedge^{n-1} V & \textit{if } \ell \textit{ is odd.} \\ \bigwedge^0 V \oplus \bigwedge^2 V \oplus \cdots \oplus \bigwedge^{n} V & \textit{if } \ell \textit{ is even.} \end{cases}$$

This is an irrep decomposition, except for $\bigwedge^{n/2} V \cong U_+ \oplus U_-$.

In some sense, the U_+ and U_- are the only things distinguishing the two infinite volume ground states ω_+ and ω_-, and they only appear when the chain is long enough $\ell \geq n/2$.

- This is a representation-theoretic symptom of the "short chain indistinguishability"!

Another index

- We can extract yet another invariant of G-gapped phases[4].
- If we cut the lattice $\mathbb{Z} = \mathbb{Z}_{(-\infty,0]} \cup \mathbb{Z}_{[1,\infty)}$, we expose some degrees of freedom of the state.
- If we then rotate the half-chain, this gives another representation of $Spin(n)$. This is a type of bulk-boundary correspondence.
- For our MPS, this is exactly Π!
- We can assign a (Borel) group cohomology index $h \in H^2(SO(n), U(1))$ to any projective rep.
    - Can show $H^2(SO(n), U(1)) \cong \mathbb{Z}_2$. There is only trivial and nontrivial projective.
- Chen-Gu-Liu-Wen 2013, Ogata 2020: SPT index h is invariant within a G-gapped phase.

[4] aka SPT index

SPT indices

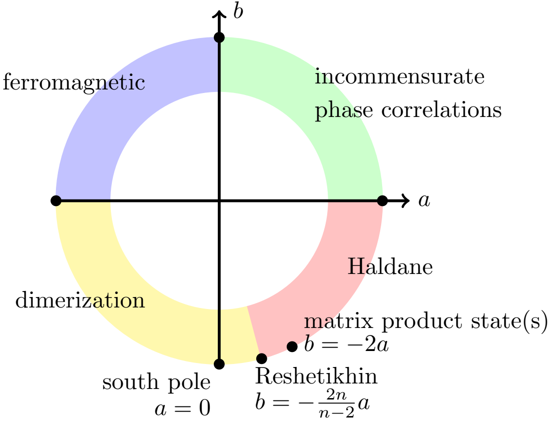
Theorem (Nachtergaele, **R**)

The south pole point has a pair of trivial SPT indices, and the MPS point has a pair of nontrivial SPT points.

Corollary

The south pole point and the MPS point occupy distinct $SO(n)$ gapped phases.

Who knows what else is hiding in this $O(n)$-chain phase diagram?



Maybe **your** PhD!