

Final

Math 121A, Fall 2025.

December 18, 2025

Unless otherwise stated, all solutions require clear, well-written justification. You may use results computed in homework or class, as long as it is clear to which result you are referring.

There are 10 problems on this final, each worth 10 points.

The following series expansions may be useful. Be mindful of their convergence.

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, & \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, & \log(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, & \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

I will also remind you of a few key conventions. Inner products:

$$\begin{aligned} \langle f, g \rangle &= \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx, & f, g &\in L^2(\mathbb{R}) \\ \langle f, g \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx, & f, g &\in L^2([-\pi, \pi]). \\ \langle \hat{f}, \hat{g} \rangle &= \sum_{k=-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k), & \hat{f}, \hat{g} &\in \ell^2(\mathbb{Z}). \end{aligned}$$

Fourier series for $L^2([-\pi, \pi])$:

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, & \hat{f}(k) &= \left\langle e^{ikx}, f \right\rangle, \\ f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, & a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, & b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx. \end{aligned}$$

Fourier transform and inverse Fourier transform:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \end{aligned}$$

Problem 1. Let $\mathcal{H}$ be an n dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ inducing the norm $\|\cdot\|$, and denote the adjoint of a linear map $A : \mathcal{H} \rightarrow \mathcal{H}$ by $A^\dagger$.

- (a) (3 points) Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a linear map. Show that if λ is an eigenvalue of A , then the complex conjugate $\bar{\lambda}$ is an eigenvalue of $A^\dagger$.
- (b) (4 points) Show that if $P : \mathcal{H} \rightarrow \mathcal{H}$ is an orthogonal projection, then for every $v \in \mathcal{H}$, the

$$\|Pv\| \leq \|v\|.$$

Hint: use the spectral theorem and that the eigenvalues of P are only 0 or 1.

- (c) (3 points) Suppose A is self-adjoint. Show that if v is an eigenvector of eigenvalue λ , and w is an eigenvector of eigenvalue $\mu \neq \lambda$, then v and w are orthogonal.

Problem 2. Consider $f \in L^2([-\pi, \pi])$ given by

$$f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi. \end{cases}$$

Expand f in a Fourier series and use Parseval's theorem to find the sum of the inverses of odd squares:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Problem 3. (a) (5 points) Fix any $h \in \mathbb{R}$. Show that the operator $\tau_h : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, defined below as convolution with a dirac delta, is unitary:

$$\tau_h f := \delta_h * f, \quad f \in L^2(\mathbb{R}).$$

(b) (5 points) By taking a Fourier transform of both sides, show that

$$\widehat{\tau_h f}(k) = e^{-ikh} \widehat{f}(k), \quad \widehat{f} \in L^2(\mathbb{R}).$$

Problem 4. Consider the linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$f(x, y, z) = (x + y, z).$$

- (a) (*2 points*) Find the matrix for f , assuming the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.
- (b) (*3 points*) Compute a basis for $\text{null}(f)$. What is its dimension?
- (c) (*3 points*) Compute a basis for $\text{ran}(f)$. What is its dimension?
- (d) (*2 points*) Is there a solution to $f(x, y, z) = (1, 1)$? If so, is it unique?

Problem 5. Use the Euler-Lagrange equations to minimize the given functional:

$$\int_{x_1}^{x_2} \frac{(y')^2}{x^3} dx.$$

As a reminder, your answer should be in the form of a function $y = g(x)$, where g may depend on some general constants.

Problem 6. (a) (4 points) Find and identify the singularities of the function

$$f(z) = \frac{1}{1 + e^z}.$$

(b) (4 points) Compute the residue of

$$\frac{\cos^2(z) - 1}{z^3} \quad \text{at } z = 0.$$

(c) (2 points) Consider the below Laurent series, which describes a function holomorphic on the punctured disc of radius $1/2$:

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{(k+1)!z^k} + \sum_{k=0}^{\infty} z^k, \quad z \in D_{1/2}(0) \setminus \{0\}.$$

Decide what type of singularity f has at $z = 0$.

Problem 7. Compute the integral

$$\int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2 + 9} dx.$$

Problem 8. Compute the integral

$$\int_{|z|=3} z^3 \exp\left(\frac{1}{z^2}\right) dz,$$

where the circle $|z| = 3$ is oriented positively.

Problem 9. Let $V = \text{ran}(f)$ where f is given by the following matrix (in the standard basis):

$$[f] = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- (a) (*5 points*) Compute an orthonormal basis for V (with respect to the usual dot product).
- (b) (*5 points*) Use this basis to compute the projection of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ onto the subspace V ,

$$\text{proj}_V \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 10. Compute the Fourier transform of the function

$$f(x) = e^{-|x|}, \quad x \in \mathbb{R}.$$

(Scratch work, please label with corresponding problem if you want it graded)

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