

① a) $\sum \frac{1}{n \log n}$ diverges by integral test:

$$\int_{100}^{\infty} \frac{1}{x \log x} dx = \int_{\log(100)}^{\infty} \frac{1}{u} du = \log(\log(x)) \Big|_{x=100}^{\infty}$$

↑
u sub

$$u = \log x$$

$$du = \frac{1}{x} dx$$

$\log(\log(x)) \xrightarrow{x \rightarrow \infty} \infty$, so this diverges
by integral test.

b) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges by comparison test:

check:

$$n=0: \frac{1}{1}$$

$$n=1: \frac{2}{1}$$

$$n=2: \frac{2 \cdot 2}{2 \cdot 1}$$

$$n=3: \frac{2 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 1}$$

$$n=4: \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{2}{1}$$

$$\frac{2}{4} \cdot \frac{2}{3} \leq \left(\frac{2}{3}\right)^2$$

Integral:

$$\frac{2^n}{n!} \leq \left(\frac{2}{3}\right)^n \frac{2}{2} \cdot \frac{2}{1}, \text{ so } \sum_{n=0}^{\infty} \frac{2^n}{n!} \leq 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

converges by geom.
series, $|\frac{2}{3}| < 1$.

c) $\sum_{n=1}^{\infty} \frac{\cos n!}{n^{1.2}}$ converges b-y comparison with p series,

Since $|\cos n!| \leq 1$ for all n ,

$$\sum_{n=1}^{\infty} \left| \frac{\cos n!}{n^{1.2}} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$$

↑ converges b-y p series,
 $p=1.2 > 1$.

(2)

$$\int_0^x \tan t \, dt = \int_0^x \frac{\sin t}{\cos t} \, dt$$

$$= \int_0^x \frac{t - \frac{t^3}{3!} + o(t^4)}{1 - \frac{t^2}{2!} + o(t^3)} \, dt$$

Geom.
series

$$\rightarrow = \int_0^x dt \left(t - \frac{t^3}{3!} + o(t^4) \right) \left(1 + \left(\frac{t^2}{2!} + o(t^3) \right) + \left(\frac{t^2}{2!} + o(t^3) \right)^2 + o(t^4) \right)$$

$$= \int_0^x \left(t - \frac{t^3}{3!} + o(t^4) \right) \left(1 + \frac{t^2}{2!} + o(t^3) \right) \, dt$$

$$= \int_0^x \left(t + \frac{t^3}{2!} - \frac{t^3}{3!} + o(t^3) \right) \, dt$$

$$= \frac{1}{2} x^2 + \left(\frac{1}{2} - \frac{1}{6} \right) \frac{1}{4} x^4 + o(x^4)$$

$$\textcircled{3} \quad f(x,y) = (x-y, x,y)$$

$$a) \quad [f] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{check: } [f] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x \\ y \end{bmatrix}$$

b) To compute the null space for f , we must solve

$$f v = 0$$

or find rref of

$$[f; 0] = \begin{bmatrix} 1 & -1 & : & 0 \\ 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix}$$

$$\text{rref } [f; 0] = \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

There are no free variables, so

$$\boxed{\begin{aligned} \text{null}(f) &= \{0\} \\ \dim(\text{null}(f)) &= 0 \end{aligned}}$$

(Alternatively: use rank-nullity. $\text{rank}(f) = 2$,

$$\text{so } 2 = \text{rank}(f) + \dim(\text{null}(f))$$

$$= 2 + 0$$

$\Rightarrow \text{null}(f)$ is trivial.)

c) we know the columns of f span $\text{ran}(f)$,
so
$$\text{ran}(f) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and we can quickly see they are linearly independent
since there is no $a \neq 0$ such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{ran}(f)$,
and $\text{rank}(f) = 2$.

d) There are no solutions to the equation

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

since

$$\text{ref} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The last row is $0=1$, so this is
inconsistent.

(4) a)

$$f v_1 = \lambda_1 v_1$$

$$f v_2 = \lambda_2 v_2$$

$$f v_3 = \lambda_3 v_3$$

The coordinates of v_i in $\mathcal{B}_{\text{eig}} = \{v_1, v_2, v_3\}$ are

$$v_1 = v_1 + 0v_2 + 0v_3 \quad [v_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = 0v_1 + v_2 + 0v_3 \Rightarrow [v_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = 0v_1 + 0v_2 + v_3 \quad [v_3] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so

$$[f] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

b) This can be read off from the matrix.

$$[f] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} \lambda_1 \neq 0 \\ \lambda_2 \neq 0 \end{matrix}$$

$$\text{ran}(f) = \text{span}\{v_1, v_2\}$$

$$\Rightarrow \boxed{\dim(\text{ran}(f)) = 2}$$

$$\text{null}(f) = \text{span}\{v_3\}$$

$$\Rightarrow \boxed{\dim(\text{null}(f)) = 1}$$

c) Observe that

$$\begin{aligned}\frac{1}{\lambda_1} [f] &= \frac{1}{\lambda_1} \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}\end{aligned}$$

check:

$$\cdot \left(\frac{1}{\lambda_1} [f] \right)^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = \frac{1}{\lambda_1} [f]$$

$$\cdot \left(\frac{1}{\lambda_1} [f] \right)^T = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = \frac{1}{\lambda_1} [f]$$

So $\frac{f}{\lambda_1}$ is indeed an orthogonal projection.

⑤

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

a) We have $M^T = M$. So by the spectral Theorem, there is an orthonormal basis of ^{for} eigenvectors at M .

b) Eigenvalues: solve

$$\det(\lambda I - M) = 0$$

$$(\lambda - 1)^2 - 4 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 4$$

$$\lambda = 1 \pm 2$$

$$\text{so } \lambda = 3, -1.$$

Eigenvectors:

$$\lambda = 3:$$

$$Mv_3 = 3v_3$$

This may be algorithmically solved. But inspection also works:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \lambda = 3, v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

By spectral theorem, we must have that the other eigenvector is orthogonal to v_3 . So

$$\lambda = -1, \quad v_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{check: } \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -v_{-1}.$$

c) we may write

$$M = S D S^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & -1 \end{bmatrix} S^{-1}$$

cols are eigenvectors

eigenvals

$$S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(from homework, or by calculation).

so

$$M^{12} = (S D S^{-1})^{12}$$
$$= S D^{12} S^{-1}$$

eigenvalues are not changed by conjugation, so

$$\text{eigs}(M^{12}) = \text{eigs}(D^{12}) = \boxed{\{3^{12}, 1\}}$$