

Midterm 1

Math 121A, Fall 2025.

Friday, September 26

Unless otherwise stated, all solutions require clear, well-written justification.

There are 5 problems on this midterm. You must solve Problems 1 and 2, and you only need to solve 2/3 of Problems 3, 4, and 5: please mark which problems you would like graded. You will not get any extra points for solving all of them.

The following series expansions may be useful. Be mindful of their convergence, which I assume you remember.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Problem 1. Use convergence tests to decide whether the following converge or diverge.

$$(a) \quad \sum_{n=100}^{\infty} \frac{1}{n \log n}, \quad (b) \quad \sum_{n=0}^{\infty} \frac{2^n}{n!}, \quad (c) \quad \sum_{n=1}^{\infty} \frac{\cos |n|}{n^{1.2}}$$

Problem 2. Use three of the useful series expansions on the front page to estimate the integral

$$\int_0^x \tan(t) dt$$

up to $o(x^4)$ (meaning your answer should be of the form $P(x) + o(x^4)$ with $P(x)$ a degree ≤ 4 polynomial). You do not need to simplify your final expression.

Problem 3. Consider the linear map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$f(x, y) = (x - y, x, y).$$

Remember that for a set of vectors to be a basis, you must check two conditions.

- (a) Find the matrix for f , assuming the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.
- (b) Compute a basis for $\text{null}(f)$. What is its dimension?
- (c) Compute a basis for $\text{ran}(f)$. What is its dimension?
- (d) Is there a solution to $f(x, y) = (1, 1, 1)$? If so, is it unique?

Problem 4. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a diagonalizable linear map.

- (a) Let v_1, v_2, v_3 be eigenvectors of f with respective eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Compute the matrix of f in the basis $\mathcal{B}_{\text{eig}} = \{v_1, v_2, v_3\}$.
- (b) Suppose that λ_1 and $\lambda_2 \neq 0$ and $\lambda_3 = 0$. Determine the dimensions of the null space and the range, $\text{null}(f)$ and $\text{ran}(f)$.
- (c) Suppose that these eigenvectors form an orthonormal¹ basis of $\mathbb{R}^3$, and suppose that $\lambda_1 = \lambda_2 \neq 0$ and $\lambda_3 = 0$.

Show in any way you prefer that $\frac{1}{\lambda_1}f$ is an orthogonal projection.

¹with respect to the usual dot product $(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1w_1 + v_2w_2 + v_3w_3$.

Problem 5. Let M be the following matrix

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

- (a) Use a theorem to deduce that we may find an orthonormal basis of $\mathbb{R}^2$ of eigenvectors of M .
- (b) Diagonalize M , i.e. compute the eigenvectors and eigenvalues of M .
- (c) What are the eigenvalues of M^{12} ? You do not need to simplify your answer.

(Scratch work, please label with corresponding problem if you want it graded)

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