

Problem 1 $P = (P_1, P_2, P_3)$

$$\text{Maximize } f(p) = -P_1 \log P_1 - P_2 \log P_2 - P_3 \log P_3$$

$$\text{subject to } g(p) = P_1 + P_2 + P_3 = 1$$

$$\text{and } P_1, P_2, P_3 \geq 0.$$

$$\frac{\partial}{\partial x} (-x \log x) = -\log x - \frac{x}{x} = -\log x - 1$$

$$\nabla f = (-\log P_1, -\log P_2, -\log P_3) - (1, 1, 1)$$

$$\nabla g = (1, 1, 1)$$

$$\text{Set } \nabla f = \lambda \nabla g$$

$$\Rightarrow -(\log P_1, \log P_2, \log P_3) = 2\lambda (1, 1, 1)$$

$$\text{so } -\log P_1 = 2\lambda \Rightarrow P_1 = e^{-2\lambda}$$

$$\text{and likewise } P_2 = P_3 = e^{-2\lambda}$$

plug into the constraint

$$3e^{-2\lambda} = 1 \Rightarrow \lambda = -\frac{1}{2} \log \frac{1}{3}$$

which means

$$P_1 = P_2 = P_3 = \frac{1}{3}$$

Problem 2

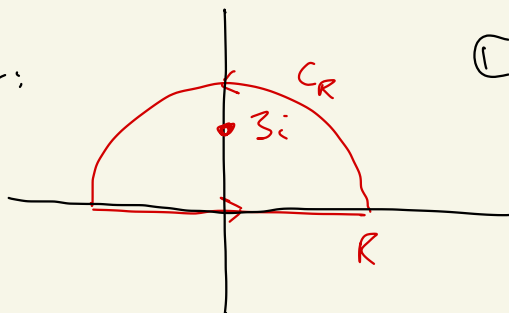
$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 9}$$

we think of this as imaginary part of

$$f(z) = \frac{e^{iz}}{z^2 + 9} = \frac{\cos z}{z^2 + 9} + i \underbrace{\frac{\sin z}{z^2 + 9}}_{\text{imaginary part}}$$

$$= \frac{e^{iz}}{(z+3i)(z-3i)} \leftarrow 2 \text{ simple poles.}$$

Choose contour:



By Residue Thm,

$$2\pi i \operatorname{Res}_{3i} f = \int_{-R}^R f + \int_{C_R} f$$

simple
pole
↓

$$\circ \operatorname{Res}_{3i} f = \lim_{z \rightarrow 3i} (z-3i) f = \lim_{z \rightarrow 3i} \frac{e^{iz}}{z+3i} = \frac{e^{-3}}{6i}$$

Now to estimate:

$$\left| \int_{C_R} f \right| \leq \int_{C_R} \left| \frac{e^{iz}}{z^2 + 9} dz \right|$$

$$= \int_0^\pi \left| \frac{e^{i(R \cos \theta + i R \sin \theta)}}{R^2 e^{i2\theta} + 9} i R e^{i\theta} \right| d\theta$$

$$= \int_0^\pi \frac{R}{R^2} \left| \frac{e^{iR \cos \theta} e^{-R \sin \theta}}{1 + \frac{9}{R^2} e^{-i2\theta}} \right| d\theta$$

larger R ,
 $1 + \frac{9}{R^2} e^{-i2\theta} \rightarrow 1$

$$\approx \int_0^\pi \frac{1}{R} |e^{iR \cos \theta} e^{-R \sin \theta}| d\theta$$

$$= \int_0^\pi \frac{1}{R} e^{-R \sin \theta} d\theta$$

Jordan
lemma

$$\leq \frac{\pi}{R^2} \xrightarrow{\lim_{R \rightarrow \infty}} 0$$

So, all together

$$\frac{2\pi i e^{-3}}{6i} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 9} dx$$

Taking Imaginary parts,

$$0 = \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 9} dx$$

← in retrospect, this is because $\frac{\sin x}{x^2 + 9}$ is an odd function.

Problem 3

a) The possible values may be seen by fixing $n \in \mathbb{Z}$ and writing

$$(-i)^i = (e^{i(\frac{3\pi}{2} + 2\pi n)})^i = e^{-3\pi/2 - 2\pi n} = \boxed{e^{-2\pi n} e^{-3\pi/2}}$$

b) By De Moivre, the solutions are

$$z_k = e^{i(\frac{\pi}{2} + 2\pi k)/4}, \quad k = 0, 1, 2, 3$$

$$\text{Ex: } (e^{i(\frac{\pi}{2} + 0)/4})^4 = e^{i\frac{\pi}{2}} = i \quad \checkmark$$

c) Since we have

$$f(z) = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z},$$

this function is holomorphic at $z=1$ since its derivative is

$$f'(z) = -\frac{1}{z^2}$$

Problem 4

a) $f(z) = \sin\left(\frac{1}{z-\pi}\right)(z-\pi)$

Laurent expansion $\rightarrow = \left(\frac{1}{z-\pi} - \frac{1}{3!}\left(\frac{1}{z-\pi}\right)^3 + \frac{1}{5!}\left(\frac{1}{z-\pi}\right)^5 + \dots\right)(z-\pi)$

$$= 1 - \frac{1}{3!}\left(\frac{1}{z-\pi}\right)^2 + \frac{1}{5!}\left(\frac{1}{z-\pi}\right)^4 + \dots$$

• since the Laurent expansion has infinitely many terms in the principal part, f has an essential singularity at $z=\pi$.

• This is the only singularity since $\sin z$ is holomorphic and $\frac{1}{z-\pi}$ is holomorphic if $z \neq \pi$.

b) $g(z) = \frac{e^{\pi z}}{(z-1)^2}$

Since $e^{\pi(1)} = e^{\pi} \neq 0$, g has a pole of order 2 at $z=1$. So, locally,

$$g(z) = \frac{a_{-2}}{(z-1)^2} + \frac{a_{-1}}{(z-1)} + \overbrace{h(z)}^{\text{holomorphic}}$$

and thus

$$\begin{aligned} \text{Res}_1 g &= \left. \frac{d}{dz} \left((z-1)^2 g(z) \right) \right|_{z=1} = \left. \frac{d}{dz} \left(e^{\pi z} \right) \right|_{z=1} \\ &= \pi e^{\pi z} \\ &= \pi e^{\pi} \end{aligned}$$

Problem 5

$$\int_0^{2\pi} \frac{d\theta}{10+8\cos\theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}.$$

↑
we did this integral in
class! see Lecture 23.

$$= \frac{1}{2} \left(\frac{2\pi}{3} \right)$$