

Midterm 2

Math 121A, Fall 2024.

October 28, 2024

Unless otherwise stated, all solutions require clear, well-written justification.

There are 5 problems on this midterm, each worth an equal amount of points. You must choose 4 out of the 5 of them. Please mark which problems you would like graded. You will not get any extra points for solving all of them, and if your marking is unclear, I will randomly choose one to not grade.

The following series expansions may be useful. Be mindful of their convergence, which I assume you remember.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Problem 1. Using Lagrange multipliers, find the largest rectangle (in area, with sides parallel to the coordinate axes and with center at the origin) which can be inscribed in the ellipse:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Problem 2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} dx.$$

- Problem 3.** (a) Find all possible values of i^i .
- (b) Find all solutions to $z^3 + 1 = 0$ (you may leave them in any form you please, so long as it is clear).
- (c) Prove or disprove: the function $f(z) = \bar{z}$ is holomorphic at $z = 1$.

Problem 4. Locate the singularities of the following function:

$$f(z) = \frac{e^{\pi/z}}{(z - \pi)^2}.$$

Classify them as removable singularities, poles (state the order), or essential singularities, and calculate the residue at each pole.

Problem 5. Evaluate the following two integrals, where the circles are oriented positively.

(a)

$$\int_{|z|=2} |z| \, dz.$$

(b)

$$\int_{|z-1|=4} \frac{e^z + \sin z}{(z - \pi)^2} dz$$

(Scratch work, please label with corresponding problem if you want it graded)

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