

① we need to check two things: $P^2 = P$ and $P^+ = P$.

$\boxed{P^2 = P}$ Let $f \in L^2(\mathbb{R})$.

$$(P^2 f)(x) = \chi(x) \int_{-\infty}^{\infty} \chi(y) (Pf)(y) dy$$

$$= \chi(x) \int_{-\infty}^{\infty} \chi(y) \chi(y) \int_{-\infty}^{\infty} \chi(z) f(z) dz dy$$

$\chi(y)^2 = \chi(y) \rightarrow$

$$= \chi(x) \int_{-\infty}^{\infty} \chi(z) f(z) dz \int_{-\infty}^{\infty} \chi(y) dy$$

$$= \chi(x) \int_{-\infty}^{\infty} \chi(z) f(z) dz \cdot 1$$

$$= (Pf)(x).$$

$\boxed{P^+ = P}$ Let $f, g \in L^2(\mathbb{R})$

$$\langle f, Pg \rangle = \int_{-\infty}^{\infty} \overline{f(x)} \int_{-\infty}^{\infty} \chi(x) \chi(y) g(y) dy dx$$

$\overline{\chi} = \chi \rightarrow$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\chi(y) \chi(x) f(x)} dx g(y) dy$$

$$= \langle Pf, g \rangle.$$



(2) (This is closely related to HW10, problem d iii)

Let us split using linearity:

$$f(\pi - |x|) = f(\pi) - f(|x|)$$

- $f(\pi) = \pi$

- $f(|x|)$: Note this function is even, so we only need a's.

$$\frac{a_0}{2} = \frac{\langle |x|, 1 \rangle}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2}$$

$$a_k = \langle |x|, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos kx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos kx dx$$

$$u = x \quad dv = \cos kx dx \\ du = dx \quad v = \frac{1}{k} \sin kx$$

Integrate
by parts

$$= \frac{2}{\pi} \left(\frac{x \sin kx}{k} \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{k} \sin kx dx \right)$$

$$= \frac{2}{\pi} \left(\frac{\pi}{k} (\sin \pi k - 0) - \frac{1}{k} \left(-\frac{1}{k} \cos kx \Big|_0^{\pi} \right) \right)$$

$$= \frac{2}{\pi k^2} (+\cos k\pi - 1)$$

$$= \begin{cases} -\frac{4}{\pi k^2} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

So:

$$\pi - |x| = \frac{\pi}{2} + \sum_{k \text{ odd}} \frac{4}{\pi k^2} \cos kx$$

③ a) Following the hint,

$$\begin{aligned}\frac{d}{ds} (\mathcal{L}f)(s) &= \frac{d}{ds} \int_0^{\infty} f(t) e^{-st} dt \\&= \int_0^{\infty} f(t) (-t) e^{-st} dt \\&= \int_0^{\infty} -t f(t) e^{-st} dt \\&= -\mathcal{L}(t f(t))(s)\end{aligned}$$

$$\begin{aligned}b) (\mathcal{L}(e^{-at} f(t)))(s) &= \int_0^{\infty} e^{-at} f(t) e^{-st} dt \\&= \int_0^{\infty} f(t) e^{-(s+a)t} dt \\&= \mathcal{L}(f(t))(s+a)\end{aligned}$$

$$\begin{aligned}c) \mathcal{L}(t e^{-\pi t} \sin t) &= -\frac{d}{ds} \mathcal{L}(e^{-\pi t} \sin t) \\&= -\frac{d}{ds} \mathcal{L}(\sin t)(s+\pi)\end{aligned}$$

using $\mathcal{L}(\sin t) = \frac{1}{s^2 + 1}$

Then $\mathcal{L}(\sin t)(s + \pi) = \frac{1}{(s + \pi)^2 + 1}$

and $-\frac{d}{ds} \left(\mathcal{L}(\sin t)(s + \pi) \right) = -2(s + \pi) \cdot \frac{-1}{((s + \pi)^2 + 1)^2}$

$$= \frac{2(s + \pi)}{((s + \pi)^2 + 1)^2}$$

④

a)

$$\mathcal{F}(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a+ik)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+ik} \right)$$

b)

$$\|g * g\|^2 \stackrel{\text{Parseval}}{=} \|\widehat{g * g}\|^2$$

convolution $\xrightarrow{\mathcal{F}}$ multiplication $\rightarrow \|\sqrt{2\pi} \hat{g} \hat{g}\|^2$

$$= \int_{-\infty}^{\infty} \sqrt{2\pi} |\hat{g}(k) \hat{g}(k)|^2 dk$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{(2\pi)^2} \left| \frac{1}{1+ik} \cdot \frac{1}{1+ik} \right|^2 dk$$

$$= \frac{\sqrt{2\pi}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{1}{|1+ik|^4} dk$$

$$= \frac{\sqrt{2\pi}}{(2\pi)^2} \frac{\pi}{2}$$

5) a) $f(x) = x$ is odd
 $g(x) = x^2$ is even,

so $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx = 0$

But by Parseval,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle = 0$$

b) Use that $g(x) = 2x = \frac{d}{dx} x^2 = \frac{d}{dx} f(x)$

$$\hat{g}(k) = \widehat{\frac{d}{dx} f}(k) = ik \hat{f}(k)$$

$$= (-1)^k \frac{2i}{k}, \quad k \in \mathbb{Z} \setminus \{0\}$$

and $\hat{g}(0) = 0$

c) Write:

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$$

$$= 2e^{-ix} + 2 + 2e^{ix}$$