

Midterm 3

Math 121A, Fall 2025.

December 3, 2025

Unless otherwise stated, all solutions require clear, well-written justification. You may use results computed in homework or class, as long as it is clear to which result you are referring.

There are 5 problems on this midterm, each worth an equal amount of points. You must choose 4 out of the 5 of them. Please mark which problems you would like graded. You will not get any extra points for solving all of them, and if your marking is unclear, I will randomly choose one to not grade.

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, & \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, & \log(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, & \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

I will also remind you of a few key conventions. Inner products:

$$\begin{aligned} \langle f, g \rangle &= \int_{-\infty}^{\infty} \overline{f(x)} g(x) dx, & f, g &\in L^2(\mathbb{R}) \\ \langle f, g \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx, & f, g &\in L^2([-\pi, \pi]). \\ \langle \hat{f}, \hat{g} \rangle &= \sum_{k=-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k), & \hat{f}, \hat{g} &\in \ell^2(\mathbb{Z}). \end{aligned}$$

Fourier series for $L^2([-\pi, \pi])$:

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}, & \hat{f}(k) &= \left\langle e^{ikx}, f \right\rangle, \\ f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, & a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, & b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx. \end{aligned}$$

Fourier transform and inverse Fourier transform:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \end{aligned}$$

Problem 1. Define a function χ by

$$\chi(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $P : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the linear map defined by

$$(Pf)(x) = \int_{-\infty}^{\infty} \chi(x)\chi(y)f(y) \, dy.$$

Show that P is an orthogonal projection.

Problem 2. Compute the trigonometric Fourier series for the function

$$f(x) = \pi - |x|, \quad -\pi < x \leq \pi$$

Problem 3. In the following, given a function $f(t), t \geq 0$, let the Laplace transform be $\mathcal{L}(f(t))(s) := F(s)$. You do not need to worry about convergence conditions on the complex number s .

- (a) (3 points) Derive the following expression for the Laplace transform, where $\mathcal{L}(f)(s) =: F(s)$:

$$\mathcal{L}(tf(t)) = -\frac{dF}{ds}$$

(Hint: take the derivative $\frac{d}{ds}\mathcal{L}[f(t)](s)$).

- (b) (3 points) Derive the following expression for the Laplace transform:

$$\mathcal{L}(e^{-at}f(t))(s) = F(s+a), \quad a \in \mathbb{C}.$$

- (c) (4 points) Use the first two parts and the known Laplace transform $(\mathcal{L}(\sin(t)))(s) = \frac{1}{s^2+1}$ to compute the Laplace transform of

$$te^{-\pi t} \sin(t).$$

Problem 4. Let $f(x) = e^{-ax}h(x)$ be a function in $L^2(\mathbb{R})$ dependent on a positive number $a > 0$, where h is the Heaviside step function:

$$h(x) := \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) (5 points) Compute the Fourier transform of f .
- (b) (5 points) Let $g(x) = e^{-x}h(x)$. Use part (a) to compute the squared $L^2(\mathbb{R})$ norm of the convolution $\|g * g\|^2$. *Hint: the following integral may be useful.*

$$\int_{-\infty}^{\infty} \frac{1}{|1 + ik|^4} dk = \frac{\pi}{2}.$$

Problem 5. In the following questions, let $\mathcal{F} : L^2([-\pi, \pi]) \rightarrow \ell^2(\mathbb{Z})$ denote the (exponential) Fourier transform, $\mathcal{F}(f) = \hat{f}$.

(a) (3 points) Let $f(x) = x$ and let $g(x) = x^2$ be functions in $L^2([-\pi, \pi])$. Find the inner product of their Fourier series $\langle \hat{f}, \hat{g} \rangle$ in any way you prefer.

(b) (4 points) On our homework we found the Fourier series of $f(x) = x^2$ to be

$$\hat{f}(k) = (-1)^k \frac{2}{k^2}, \quad k \in \mathbb{Z}.$$

Use this to find the Fourier series of $g(x) = 2x$.

(c) (3 points) Compute the inverse Fourier transform of the sequence $\hat{f} \in \ell^2(\mathbb{Z})$, where

$$\hat{f}(k) = \begin{cases} 2 & |k| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(Scratch work, please label with corresponding problem if you want it graded)

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